**Razmyslov and quivers**
**Notes for our seminar**
**Lex Schrijver**

# 1. Razmyslov

(Razmyslov [3], cf. Procesi [2].)

If $A, B, C, D$ are $2 \times 2$ matrices, then

$$
\begin{aligned}
(1)\qquad & 2\mathrm{tr}(ABCD) = +(\mathrm{tr}(AB)\mathrm{tr}(CD) + \mathrm{tr}(BC)\mathrm{tr}(DA)) \\
& -(\mathrm{tr}(AB)\mathrm{tr}(C)\mathrm{tr}(D)+\mathrm{tr}(BC)\mathrm{tr}(A)\mathrm{tr}(D)+\mathrm{tr}(CD)\mathrm{tr}(A)\mathrm{tr}(B)+\mathrm{tr}(DA)\mathrm{tr}(B)\mathrm{tr}(C)) \\
& +(\mathrm{tr}(ABC)\mathrm{tr}(D) + \mathrm{tr}(BCD)\mathrm{tr}(A) + \mathrm{tr}(CDA)\mathrm{tr}(B) + \mathrm{tr}(DAB)\mathrm{tr}(C)) \\
& -\mathrm{tr}(AC)\mathrm{tr}(BD) \\
& +\mathrm{tr}(A)\mathrm{tr}(B)\mathrm{tr}(C)\mathrm{tr}(D).
\end{aligned}
$$

On the other hand, if $A = E_{1,1} - E_{2,2}$, $B = E_{1,2}$, $C = E_{2,1}$, then $\mathrm{tr}(ABC) = 1$, while $tr(A) = \mathrm{tr}(B) = \mathrm{tr}(C) = 0$. So $\mathrm{tr}(ABC)$ is not in the algebra generated by smaller traces (in fact, over any characteristic).

Let $n, m \in \mathbb{N}$. Consider the polynomial $\mathrm{tr}(X_1 \cdots X_m)$ on $\bigoplus_{i=1}^m \mathbb{R}^{n\times n}$, where $X_1, \ldots, X_m \in \mathbb{R}^{n\times n}$. Let $m_n$ be the largest $m$ for which $\mathrm{tr}(X_1 \ldots X_m)$ is *not* in the algebra generated by traces of products of less than $m$ different $X_i$ (over characteristic 0).

**Question:** What is $m_n$ ?

**Theorem 1** (Razmyslov)**.** *$m_n \leq n^2$.*

Known: $m_n \geq \binom{n+1}{2}$, with equality if $n \leq 3$.

For $m \in \mathbb{N}$, let $S'_m$ be the set of permutations in $S_m$ with at least two orbits. For any $\pi \in S_m$, let $\Omega_\pi$ be the set of orbits of $\pi$. For any $\omega \in \Omega_\pi$, let $i_\omega$ be the smallest element of $\omega$.

**Proposition 1.** *$m > m_n$ if and only if there exists $\lambda : S'_m \to \mathbb{R}$ such that*

$$
(2)\qquad \mathrm{tr}(X_1 \cdots X_m) = \sum_{\pi\in S'_m} \lambda_\pi \prod_{\omega\in\Omega_\pi} \mathrm{tr}(\prod_{j=1}^{|\omega|} X_{\pi^j(i_\omega)}).
$$

**Proof.** In writing $\mathrm{tr}(X_1 \cdots X_m)$ as a linear combination of products of $\mathrm{tr}(X_{i_1} \cdots X_{i_k})$ with $k < m$ and $i_1, \ldots, i_k \in [m]$, we can restrict these products to those in which each $X_i$ occurs exactly once (by homogeneity). That is, the summands can be described as the terms in the right hand side of (2). ∎

Let $A_{n,m}$ be the $S_m \times ([n]^m)^2$ matrix with

$$
(3)\qquad (A_{n,m})_{\pi,(f,g)} = \delta_{g,f\pi},
$$

for $\pi \in S_m$ and $f, g \in [n]^m$. Let $A'_{n,m}$ be the $S'_m \times ([n]^m)^2$ submatrix of $A_{n,m}$.

**Proposition 2.** *$m > m_n$ if and only if* $\mathrm{rank}(A'_{n,m}) = \mathrm{rank}(A_{n,m})$.

**Proof.** *Necessity.* Suppose $m > m_n$, and let $\lambda : S'_m \to \mathbb{R}$ satisfy (2). By symmetry, it suffices to show that $\lambda A'_{n,m}$ is equal to row $\gamma$ of $A_{n,m}$, where $\gamma$(i)$=i+1$ for $i = 1, \ldots, m$ (mod $m$).

Consider a column index $(f, g) \in ([n]^m)^2$. Define $U_k := E_{f(k),g(k)}$ for $k = 1, \ldots, m$. (Here $E_{i,j}$ is the $n \times n$ matrix with 1 in position $i, j$ and 0 elsewhere.) Then

$$(A_{n,m})_{\gamma,(f,g)} = \delta_{g,f\gamma} = \mathrm{tr}(U_1 \cdots U_m) = \sum_{\pi \in S'_m} \lambda_\pi \prod_{\omega \in \Omega_\pi} \mathrm{tr}(\prod_{j=1}^{|\omega|} U_{\pi^j(i_\omega)}) = \sum_{\pi \in S'_m} \lambda_\pi \delta_{g,f\pi} = \sum_{\pi \in S'_m} \lambda_\pi (A_{n,m})_{f,g}. \tag{4}$$

This proves necessity.

*Sufficiency.* Reverse the above argumentation. Note that in proving (2) we can assume that each $X_k$ is equal to $E_{i,j}$ for some $i, j$. ∎

Let $M_{n,m}$ be the $S_m \times S_m$ matrix with

$$(M_{n,m})_{\pi,\rho} := n^{o(\pi\rho^{-1})}, \tag{5}$$

for $\pi, \rho \in S_m$ (where $o(\pi\rho^{-1}) = |\Omega_{\pi\rho^{-1}}|$). Let $M'_{n,m}$ be the $S'_m \times S'_m$ submatrix of $M_{n,m}$.

**Proposition 3.** *$m > m_n$ if and only if* $\mathrm{rank}(M'_{n,m}) = \mathrm{rank}(M_{n,m})$.

**Proof.** $M_{n,m} = A_{n,m} A_{n,m}^\mathsf{T}$. Indeed, $(A_{n,m} A_{n,m}^\mathsf{T})_{\pi,\rho}$ is equal to the number of $(f, g) \in ([n]^m)^2$ satisfying $g = f\pi$ and $g = f\rho$. Hence it is equal to the number of $f \in [n]^m$ satisfying $f\pi = f\rho$, i.e., $f\pi\rho^{-1} = f$. This number is equal to $n^{o(\pi\rho^{-1})}$. ∎

**Note:** $\mathrm{rank}(A_{n,m}) = \mathrm{rank}(M_{n,m}) = \sum_{\substack{\lambda \vdash m \\ \mathrm{height}(\lambda) \leq n}} (f^\lambda)^2$ (cf. [4]).

| $n$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $\mathrm{rank}(M_{n,m})$ | 1 | 14 | 2761 | 19318688 | 8636422912277 |

, where $m = \binom{n+1}{2} + 1$.

## 2. Quivers

A *quiver* is a directed graph $D = (V, A)$. For each $v$, let $L_v$ be a linear space, and for each $a = (u, v) \in A$, let $M_a := \mathrm{Hom}(L_u, L_v) \cong L_u^* \otimes L_v$. Define

$$\mathcal{R} := \bigoplus_{a=(u,v) \in A} M_a \quad \text{and} \quad G := \prod_{v \in V} \mathrm{GL}(L_v). \tag{6}$$

Any element of $\mathcal{R}$ is called a *representation* of $D$. (The representations are in one-to-one relation with the $\mathbb{F}Q$-modules, where $\mathbb{F}Q$ is the *path algebra* of $Q$, which is spanned by all walks $W = (v_0, a_1, v_1, a_2, \ldots, v_t)$ in $Q$, with concatenation $W_1 W_2$ as product if the end vertex of $W_1$ equals the start vertex of $W_2$, and product $= 0$ otherwise. Note that $\sum_{v\in V}(v)$ is the unit of this algebra.)

The group $G$ acts on $\mathcal{R}$ by

$$\sum_{a=(u,v)\in A} R_a \mapsto \sum_{a=(u,v)\in A} g_u R_a g_v^{-1} \tag{7}$$

for $R = \sum_{a\in A} R_a \in \mathcal{R}$, $R_a \in M_a$ $(a \in A)$, and $g \in G$.

Le Bruyn and Procesi [1]:

**Theorem 2.** *$\mathcal{O}(\mathcal{R})^G$ is generated by the polynomials*

$$\operatorname{tr}(\prod_{j=1}^{k} R_{a_j}) \tag{8}$$

*where $a_1, \ldots, a_k$ are the consecutive edges of a closed walk in $D$ traversing each vertex $v$ at most $m_{\dim(L_v)}$ times.*

**Proof.** Choose

$$\begin{aligned} p \in \mathcal{O}(\mathcal{R})^G = \mathcal{O}(\bigoplus_{a\in A} M_a)^G = \big(\prod_{a\in A}\mathcal{O}(M_a)\big)^G = \big(\prod_{a\in A}\bigoplus_{d\in\mathbb{N}}\mathcal{O}(M_a)_d\big)^G = \\ \big(\bigoplus_{d:A\to\mathbb{N}}\prod_{a\in A}\mathcal{O}(M_a)_{d_a}\big)^G = \bigoplus_{d:A\to\mathbb{N}}\big(\prod_{a\in A}\mathcal{O}(M_a)_{d_a}\big)^G. \end{aligned} \tag{9}$$

So we can assume that there exists $d : A \to \mathbb{N}$ with

$$p \in \prod_{a\in A}\mathcal{O}(M_a)_{d_a}. \tag{10}$$

I. We can assume that $d_a = 1$ for each $a \in A$. For let $D' = (V, A')$ be obtained from $D$ by replacing each edge $a$ by $d_a$ parallel edges. Indeed, $p$ is a linear combination of polynomials $\prod_{a\in A} p_a$, with $p_a \in \mathcal{O}(M_a)_{d_a} = (M_a^*)^{d_a}$. So

$$p = \sum_{j=1}^{K}\prod_{a\in A}\prod_{i=1}^{d_a} f_{a,i}^{(j)}, \tag{11}$$

for some $K \in \mathbb{N}$ and some $f_{a,i}^{(j)} \in M_a^*$ (for all $j = 1, \ldots, K$, $a \in A$, $i = 1, \ldots, d_a$). Let

$$\mathcal{R}' := \bigoplus_{a'=(u,v)\in A'} L_u^* \otimes L_v. \tag{12}$$

Let $p' \in \mathcal{O}(\mathcal{R}')$ be defined by

$$p'(R') = \sum_{j=1}^{K}\prod_{a\in A}\prod_{i=1}^{d_a} f_{a,i}^{(j)}(R'_{a_i}), \tag{13}$$

for $R' \in \mathcal{R}'$, where $a_i$ denotes the $i$-th parallel edge of $a$.

Consider $\tilde{p} := \rho_G(p')$, where $\rho_G$ is the Reynolds operator. If we know the theorem for $\tilde{p}$, we know it for $p$: let $\varphi : \mathcal{R} \to \mathcal{R}'$ be defined by $\varphi(R)_{a_i} = R_a$ for each $a \in A$ and $i = 1, \ldots, d_a$. Then $p = p'\varphi$, hence, as $\varphi$ is $G$-equivariant, $\tilde{p}\varphi = \rho_G(p')\varphi = \rho_G(p'\varphi) = \rho_G(p) = p$.

II. Since $d_a = 1$ for all $a \in A$, there exists $P \in \bigotimes_{a\in A} M_a$ such that for all $R \in \mathcal{R}$:

$$p(R) = \langle P, \bigotimes_{a\in A} R_a \rangle. \tag{14}$$

As $p$ is $G$-invariant, we can assume that $P$ is $G$-invariant. Hence

$$\begin{aligned} P \in \big(\bigotimes_{a\in A} M_a\big)^G &= \big(\bigotimes_{a=(u,v)\in A} (L_u^* \otimes L_v)\big)^G \cong \big(\bigotimes_{v\in V} \big( \bigotimes_{a\in\delta^{\mathrm{out}}(v)} L_v^* \otimes \bigotimes_{a\in\delta^{\mathrm{in}}(v)} L_v\big)\big)^G \\ &= \bigotimes_{v\in V} \big( \bigotimes_{a\in\delta^{\mathrm{out}}(v)} L_v^* \otimes \bigotimes_{a\in\delta^{\mathrm{in}}(v)} L_v\big)^{\mathrm{GL}(L_v)} \overset{\mathrm{FFT}}{=} \bigotimes_{v\in V} \langle t_{L_v,\pi_v} \mid \pi_v : \delta^{\mathrm{in}}(v) \leftrightarrow \delta^{\mathrm{out}}(v)\rangle. \end{aligned} \tag{15}$$

Here $t_{L_v,\pi_v}$ is the element of $L_v^{*\otimes\delta^{\mathrm{out}}(v)} \otimes L_v^{\otimes\delta^{\mathrm{in}}(v)} = \mathrm{Hom}(L_v^{\otimes\delta^{\mathrm{out}}(v)}, L_v^{\otimes\delta^{\mathrm{in}}(v)})$ with

$$t_{L_v,\pi_v} : \bigotimes_{a\in\delta^{\mathrm{out}}(v)} x_a \quad \mapsto \quad \bigotimes_{a\in\delta^{\mathrm{in}}(v)} x_{\pi_v(a)}, \tag{16}$$

for any $x : \delta^{\mathrm{out}}(v) \to L_v$.

The last expression in (15) implies that $P$ is a linear combination of tensors $\bigotimes_{v\in V} t_{L_v,\pi_v}$ that are obtained by choosing for each $v \in V$ a bijection $\pi_v : \delta^{\mathrm{in}}(v) \to \delta^{\mathrm{out}}(v)$. For each such choice, there exist closed walks $W_1, \ldots, W_l$ partitioning $A$ such that for each $a = (u, v) \in A$, the edges $a$ and $\pi_v(a)$ occur consecutively in some walk $W_i$ (circularly). Hence the term is equal to the product over $i = 1, \ldots, l$ of $\mathrm{tr}(R_{a_1} \cdots R_{a_k})$ where $a_1 \ldots a_k$ are the consecutive edges in walk $W_i$.

Now consider any closed walk $W$ traversing some vertex $v$ $m$ times with $m > m_{\dim(L_v)}$. Split $W$ into $v - v$ walks $W_1, \ldots, W_m$. Let $Y_i := \prod_{a\in W_i} R_a$, for $i = 1, \ldots, m$. So $Y_i \in \mathrm{End}(L_v)$. By definition of $m_{\dim(L_v)}$, $\mathrm{tr}(Y_1 \cdots Y_m)$ is in the algebra generated by traces of less than $m$ products of different $Y_i$. Each corresponds to a walk of length less than $m$. ∎

# References


[1] L. Le Bruyn, C. Procesi, Semisimple representations of quivers, *Transactions of the American Mathematical Society* 317 (1990) 585–598.

[2] C. Procesi, The invariant theory of $n \times n$ matrices, *Advances in Mathematics* 19 (1976) 306–381.

[3] Ju.P. Razmyslov, Trace identities of full matrix algebras over a field of characteristic zero [in Russian], *Izvestiya Akademii Nauk SSSR. Seriya Matematicheskaya* 38 (1974) 723–756 [English translation: *Mathematics of the USSR. Izvestija* 8 (1974) 727–760].

[4] A. Schrijver, Characterizing partition functions of the edge-coloring model by rank growth, *Journal of Combinatorial Theory, Series A* 136 (2015) 164–173.