Is each Mader matroid a gammoid?

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A Mader matroid is obtained as follows. Let $G = (V, E)$ be an undirected graph and let $S_1, \ldots, S_k$ be disjoint subsets of V. Define $S := S_1 \cup \cdots \cup S_k$. Let $\mathcal{I}$ be the collection of all subsets I of S with the property that there exist disjoint paths $P_1, \ldots, P_t$ such that each P_j has its ends in different sets S_i and has no internal vertices in S and such that I is a subset of the set of ends of the P_j. Then $\mathcal{I}$ is the collection of independent sets of a matroid, which we call a *Mader matroid*. The fact that it is a matroid follows from the method given in Schrijver [6], where a short proof is given of the theorem of Mader [4] on the maximum number of these paths.

If $k = 2$, we call the Mader matroid also a *Menger matroid*. The *matching matroids* are the special case of Mader matroids where $S_1, \ldots, S_k$ are all singleton subsets of V. (Matching matroids were introduced by Edmonds and Fulkerson [1], and are special cases of transversal matroids.)

The question is how Mader matroids relate to known classes of matroids. The class of gammoids seems close to Mader matroids. Recall that a gammoid is obtained as follows. Let $D = (V, A)$ be a directed graph and let S and T be disjoint subsets of V. Let $\mathcal{I}$ be the collection of subsets I of S such that there exist $|I|$ disjoint directed paths from I to T. Then $(S, \mathcal{I})$ is a matroid, called a *gammoid*.

(1) Is each Mader matroid a gammoid?

What can be proved is that each *Menger* matroid is a gammoid. More precisely:

Theorem 1. *A matroid is a gammoid if and only if it is a contraction of a Menger matroid.*

Proof. To see necessity, each gammoid is the contraction of a transversal matroid (Ingleton and Piff [2]), hence it suffices to show that each transversal matroid $M = (S, \mathcal{I})$ is a contraction of a Menger matroid. We can assume that the transversal matroid is obtained from a bipartite graph G with colour classes S and T, such that the independent sets of M are the subsets of S covered by some matching in G, and that G has a matching of size $|T|$. Let M' be the Menger matroid on $S \cup T$ obtained from G by taking S and T as classes. Then contracting T in M' gives M.

To prove sufficiency, it suffices to show that each Menger matroid is a gammoid (as the class of gammoids is closed under contractions). Let $G = (V, E)$ be an undirected graph, and let S_1 and S_2 be disjoint subsets of V. Let M be the Menger matroid obtained this way. So a subset B of $S_1 \cup S_2$ is a base of M if and only if there exists a maximum collection of disjoint paths in G from S_1 to S_2, with no intermediate vertices in $S_1 \cup S_2$, such that B is the set of ends of these paths. We can assume that neither S_1 nor S_2 spans an edge of G.

Let $D = (V, A)$ be the directed graph obtained from G by orienting each edge incident with S_1 away from S_1 and by orienting each edge incident with S_2 towards S_2, and by

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replacing each remaining edge e by two oppositely oriented arcs connecting the ends of e. So a subset B of $S_1 \cup S_2$ is a base of M if and only if there exists a maximum collection of disjoint directed paths in D from S_1 to S_2, with no intermediate vertices in $S_1 \cup S_2$, such that B is the set of ends of these paths.

Derive an undirected graph $\tilde{G}$ from D as follows. Replace each vertex $v \notin S_1 \cup S_2$ by two vertices, v' and v''. For $v \in S_1$ define $v' := v$ and for $v \in S_2$ define $v'' := v$. Replace each arc (u, v) by an edge $u'v''$. Moreover, for each $v \in V \setminus (S_1 \cup S_2)$, make an edge $v'v''$. This makes the undirected graph $\tilde{G}$. Then for any subset I of $S_1 \cup S_2$ one has, by a well-known argument:

(2) D contains disjoint directed paths from S_1 to S_2, such that I is the collection of the ends $\iff$ $\tilde{G}$ contains a matching N which covers all vertices except those in $(S_1 \cup S_2) \setminus I$.

So a subset B of $S_1 \cup S_2$ is a base of M if and only if $\tilde{G}$ has a maximum-size matching N which covers all vertices except those in $(S_1 \cup S_2) \setminus B$. So M is the matroid obtained from the matching matroid of $\tilde{G}$ by contracting all vertices not in $S_1 \cup S_2$. As each matching matroid is a transversal matroid (Edmonds and Fulkerson [1]), this proves that each Menger matroid is the contraction of a transversal matroid, and hence is a gammoid (Ingleton and Piff [2]). ∎

By the work of Edmonds and Fulkerson [1] and Ingleton and Piff [2], the class of gammoids is equal to the class of contractions of matching matroids. So contractions of Menger matroids and those of matching matroids (two special cases of Mader matroids) coincide.

A question related to (1) is:

(3) Is each Mader matroid linear?

By the results of Piff and Welsh [5], gammoids are representable over all large enough fields. Hence a positive answer to question (1) implies a positive answer to question (3). The constructions given by Lovász [3] suggest a positive answer to (3).

References

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