Semidefinite functions on categories

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Contents

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Abstract

Freedman, Lovász and Schrijver characterized graph parameters that can be represented as the (weighted) number of homomorphisms into a fixed graph. Several extensions of this result have been proved. We use the framework of categories to prove a general theorem of this kind. Similarly as previous resuts, the characterization uses certain infinite matrices, called *connection matrices*, which are required to be positive semidefinite.

1 Introduction

For two finite graphs F and G, let $\hom(F, G)$ denote the number of homomorphisms $F \to G$. The definition can be extended to weighted graphs. In [7] graph parameters of the form $\hom(\cdot, G)$, defined on finite multigraphs, were characterized, where G is a fixed weighted graph. Several variants of this result have been obtained, characterizing graph parameters $\hom(\cdot, G)$ where all nodeweights of G are 1 [16], such graph parameters defined on simple graphs [13] etc. These characterizations involve certain infinite matrices, called *connection matrices*, which are required to be positive semidefinite and to satisfy a condition on their rank. The results can be extended to directed graphs, hypergraphs etc.

The goal of this paper is to use the framework of categories to prove a general theorem of this kind. Let $\mathcal{C}$ be a category. We need to assume that it satisfies a number of natural conditions C1-C4 below, but for the statement of the main theorem we only need that it is locally finite, it has pullbacks, and it contains a terminal object t. In particular, every two objects a and b have a direct product $a \times b$. We denote by $\mathcal{C}(a, b)$ the set of morphisms from a to b.

Let f be a real valued function defined on the objects, invariant under isomorphism. We say that f is *multiplicative*, if $f(a \times b) = f(a)f(b)$ for any two objects a and b. For every object a, we define a (possibly infinite) symmetric matrix $N(f, a)$, whose rows and columns are indexed by the morphisms into a, and whose entry in row α and column β is $f(p(\alpha, \beta))$, where

$p(\alpha, \beta)$ is the object where the pullback of (α, β) starts (this is well defined up to isomorphism).

Theorem 1 *Let $\mathcal{C}$ be a category satisfying conditions* C1-C4 *below. Let f be a function defined on the objects, invariant under isomorphism. Then $f = |\mathcal{C}(b, .)|$ for some object b if and only if the following conditions are fulfilled:* (F1) $f(t) = 1$, (F2) *f is multiplicative, and* (F3) *$N(f, a)$ is positive semidefinite for every object a.*

We note that if there is a monomorphism from a to b, then $N(f, a)$ is a submatrix of $N(f, b)$. Thus it would be enough to require the semidefiniteness condition for an appropriate subset K of objects such that every object has a monomorphism into some $k \in K$ (we call such a set K *cofinal*). Since $a \times t$ is isomorphic with a, condition (F1) follows from (F2) unless f is identically 0.

Let us mention a corollary.

Corollary 2 *Conditions* (F1)–(F3) *of the theorem imply that (a) the values of f are non-negative integers, (b) the rank of $N(f, a)$ is at most $\mathcal{C}(b, a)$.*

Part (a) contrasts this result with the results of [7, 16], where (thanks to the weights) the function values can be arbitrary. An analogue of (b) must be imposed as a condition e.g. in the characterization in [7], while in this setup it follows from the other assumptions.

2 Preliminaries

2.1 Conditions on the category

Let $\mathcal{C}$ be a category (for basic definitions and facts, see e.g. [1]). For two objects $a, b \in \mathrm{Ob}(\mathcal{C})$, we denote by $\mathcal{C}(a, b)$ the set of morphisms $a \to b$. For $\alpha \in \mathcal{C}(a, b)$, we set $t(\alpha) := a$ and $h(\alpha) := b$. Let $\mathcal{C}_a$ denote the set of morphisms with $h(\alpha) = a$. We denote by $\mathcal{C}^{\mathrm{mon}}(a, b)$ and by $\mathcal{C}_a^{\mathrm{mon}}$ the set of monomorphisms in $\mathcal{C}(a, b)$ and $\mathcal{C}_a$, respectively.

We make the following assumptions about our category.

C1 (a) $\mathcal{C}$ is locally finite, i.e., $\mathcal{C}(a, b)$ is finite for all a, b.

(b) For every object a there is only a finite number of nonisomorphic objects that have a monomorphism into a.

C2 (a) $\mathcal{C}$ has pullbacks.

(b) $\mathcal{C}$ has a terminal object t, into which every object has a unique morphism (which can be thought of as the pullback of the empty set of morphisms).

C3 Every morphism is the product of an epimorphism and a monomorphism.

C4 The category has an object such that the set of its direct powers is cofinal (we call such an object a *generator*).

For every object a, we introduce an equivalence relation on $\mathcal{C}_a$ by $\alpha \equiv \beta$ if and only if $\beta = \gamma\alpha$ for some isomorphism γ. We say that α and β are *left-isomorphic*. We denote by $[\alpha]$ the equivalence class of α, and by $\widehat{\mathcal{C}_a}$, the set of equivalence classes in $\mathcal{C}_a$.

Recall that for two morphisms $\alpha \in \mathcal{C}(a,c)$ and $\beta \in \mathcal{C}(b,c)$, a pair of morphisms $\alpha' \in \mathcal{C}(d,a)$ and $\beta' \in \mathcal{C}(d,b)$ is called a *pullback* of (α,β) if $\alpha'\alpha = \beta'\beta$, and whenever $\xi \in \mathcal{C}(e,a)$ and $\zeta \in \mathcal{C}(e,b)$ are two morphisms such that $\xi\alpha = \zeta\beta$, then there is a unique morphism $\eta \in \mathcal{C}(e,d)$ such that $\eta\alpha' = \xi$ and $\eta\beta' = \zeta$. We also call α' a *pullback of* β *along* α.

In terms of α and β, we write

$$p(\alpha,\beta) := d, \quad \beta^*(\alpha) := \beta', \qquad \alpha^*(\beta) := \alpha', \qquad \alpha \times \beta := \alpha'\alpha = \beta'\beta.$$

(This strange notation will be convenient later on.)

It is well known and easy to check that for $\alpha,\beta \in \mathcal{C}_a$, $[\alpha^*(\beta)]$ only depends on $[\beta]$, and $[\alpha \times \beta]$ only depends on $[\alpha]$ and $[\beta]$. The object $p(\alpha,\beta)$ is determined up to isomorphism. Furthermore, if $[\alpha_1] = [\alpha_2]$, then $[\beta^*(\alpha_1)] = [\beta^*(\alpha_2)]$ and $[\beta \times \alpha_1] = [\beta \times \alpha_2]$. So the operation $\times$ is well defined on equivalence classes of morphisms. It is also clear that if $\alpha_1, \alpha_2 \in \mathcal{C}(a,b)$, $\varphi \in \mathcal{C}(b,c)$, and $[\alpha_1] = [\alpha_2]$, then $[\alpha_1\varphi] = [\alpha_2\varphi]$. This defines $[\alpha] \times [\beta] := [\alpha \times \beta]$. It is easy to see that the operation $\times$ on $\widehat{\mathcal{C}_a}$ is associative and commutative.

We say that the category has pullbacks (condition C2(a)) if every pair of morphisms into the same object has a pullback. A *direct product* $a \times b$ of two objects is any object of the form $p(\alpha,\beta)$, where α and β are the unique morphisms of a and b into the terminal object t. This is uniquely determined up to isomorphism.

2.2 Examples

Example 1 The category of finite simple graphs with loops (where morphisms are homomorphisms, i.e., adjacency-preserving maps) satisfies these assumptions. Conditions C1 and C3 are trivial.

The terminal object in C2(b) is the single node with a loop, while any complete graph on 2 or more nodes with loops can serve as a generator object as in C4. To construct the pullback of two homomorphisms $\alpha: a \to c$ and $\beta: b \to c$, take the direct (categorial) product d of the two graphs a and b, together with its projections π_a and π_b onto a and b, respectively, and take the subgraph d' of d induced by those nodes v for which $(\pi_a\alpha)(v) = (\pi_b\beta)(v)$, together with the restrictions of π_a and π_b onto d'.

The cofinal set mentioned in the remark after the Theorem can be the set of all complete graphs with loops at all nodes, in which case the conditions of Theorem 1 are exactly the conditions given in [11].

Example 2 Reversing the arrows in the category of finite simple graphs with loops (Example 1) gives another category satisfying the assumptions.

Conditions C1 and C3 are again trivial. The terminal object in C2(b) is the empty graph, a generator object is the single node without a loop.

In this dual setting, we have to construct the pushout of two homomorphisms $\alpha: c \to a$) and $\beta: c \to b$). This can be done by taking the disjoint union of the two graphs a and b, and identifying those nodes that are the images of one and the same node of c. This is just the construction of the connection matrix given in [11]. The cofinal set mentioned in the remark after the Theorem can be the set of all graphs with no edges, in which case the conditions of Theorem 1 are exactly the conditions given in [11] for this dual setting.

We note that the conditions are very similar to those in [7], except that there the graphs cannot have loops and the matrices are indexed by monomorphisms only. As a consequence, the characterization concerns homomorphism numbers into weighted graphs, which is an extension not considered in this paper. Razborov's "flag algebras" [15] are essentially subalgebras of the algebras $\mathcal{C}_a$ below (with arrows reversed), generated by induced embeddings of a fixed graph into all other graphs.

These examples can be extended to simplicial maps between simplicial complexes, homomorphisms between directed graphs, hypergraphs, etc.

2.3 Some simple properties of the category

We state some easy consequences of these assumptions. It is easy to see that condition C1 implies:

Lemma 3 *(a) Every monomorphism [epimorphism] $\mu \in \mathcal{C}(a,a)$ is an isomorphism.*

(b) If both $\mathcal{C}(a,b)$ and $\mathcal{C}(b,a)$ contain monomorphisms [epimorphisms], then a is isomorphic to b.

The operations introduced above satisfy some useful identities.

Lemma 4 (a) *Let $\alpha \in \mathcal{C}(a,b)$, $\beta \in \mathcal{C}(b,c)$ and $\varphi \in \mathcal{C}(d,c)$. Let (β', φ') be a pullback of (β', φ'), and let (α', β'') be a pullback of (α, β'). Then $(\alpha', \beta''\varphi')$ is a pullback of $(\alpha\beta, \varphi)$.*

(b) *Let $\alpha_1, \alpha_2 \in \mathcal{C}_a$ and $\varphi \in \mathcal{C}(b,a)$. Then $[\varphi^*(\alpha_1 \times \alpha_2)] = [\varphi^*(\alpha_1) \times \varphi^*(\alpha_2)]$.*

(c) *Let $\alpha_1, \alpha_2 \in \mathcal{C}_a$ and $\varphi \in \mathcal{C}(a,b)$. If φ is a monomorphism, then $[(\alpha_1 \times \alpha_2)\varphi] = [(\alpha_1\varphi) \times (\alpha_2\varphi)]$.*

Proof. The proofs of these identities is similar, and we only prove (b) and (c). We fix a particular choice of the pullbacks.

The first identity follows by the following computation:

$$\begin{aligned}\varphi^*(\alpha_1 \times \alpha_2) &= \varphi^*(\alpha_2^*(\alpha_1)\alpha_2) = (\alpha_2^*(\varphi))^*(\alpha_2^*(\alpha_1))\varphi^*(\alpha_2) \\ &= (\alpha_2^*(\varphi)\alpha_2)^*(\alpha_1)\varphi^*(\alpha_2) = (\varphi^*(\alpha_2)\varphi)^*(\alpha_1)\varphi^*(\alpha_2) \\ &= (\varphi^*(\alpha_2))^*(\varphi^*(\alpha_1))\varphi^*(\alpha_2) = \varphi^*(\alpha_1) \times \varphi^*(\alpha_2).\end{aligned}$$

(Here we used (a).)

To prove (c), let $\alpha_1 \in \mathcal{C}(c_i, a)$, and $\alpha_1 \times \alpha_2 \in \mathcal{C}(d,a)$. We want to prove that $(\alpha_1^*(\alpha_2), \alpha_2^*(\alpha_1))$ is a pullback of $(\alpha_1\varphi, \alpha_2\varphi)$. Let e be any object and let $\gamma_i \in \mathcal{C}(e, c_i)$ be morphisms such that $\gamma_1\alpha_1\varphi = \gamma_2\alpha_2\varphi$. Since φ is a monomorphism, this implies that $\gamma_1\alpha_1 = \gamma_2\alpha_2$. Since $\alpha_2^*(\alpha_1) \in \mathcal{C}(d, c_1)$ and $\alpha_1^*(\alpha_2) \in \mathcal{C}(d, c_2)$ form a pullback of (α_1, α_2), it follows that there is a morphism $\psi \in \mathcal{C}(e,d)$ such that $\gamma_1 = \psi\alpha_1^*(\alpha_2)$ and $\gamma_2 = \psi\alpha_2^*(\alpha_1)$. This proves the assertion. □

For each object a, the operation $\times$ defines a semigroup on $\widehat{\mathcal{C}}_a$. Let $\mathcal{G}_a$ denote its semigroup algebra. If $\varphi: a \to b$ is any morphism, then $\alpha \mapsto \alpha\varphi$ extends to a linear map $\mathcal{G}_a \to \mathcal{G}_b$, which we denote by $x \mapsto x\varphi$. The map $\beta \mapsto \varphi^*(\beta)$ extends to a linear map $\mathcal{G}_b \to \mathcal{G}_a$, which we denote by $x \mapsto x\varphi^*$.

Lemma 5 *Let a, b_1, b_2 be objects, $\varphi_i \in \mathcal{C}(b_i, a)$, and let (η_1, η_2) be a pullback of (φ_2, φ_2). Let $x_i \in \mathcal{G}_{b_i}$, then $x_1\varphi_1 \times x_2\varphi_2 = (x_1\eta_1^* \times x_2\eta_2^*)(\varphi_1 \times \varphi_2)$.*

Proof. It suffices to prove this for the case when $x_i = [\beta_i]$ for some $\beta_i \in \mathcal{C}_{b_i}$. Then the equation follows by applying Lemma 4(a) twice. $\square$

3 Factoring by f

Let $f : \mathcal{C} \to \mathbb{R}$ be any function invariant under isomorphism. It will be convenient to extend it to morphisms, and define $f(\varphi) = f(t(a))$. Clearly, this extension is invariant under left-isomorphism of morphisms. We can extend f to the algebras $\mathcal{G}_a$ linearly. It follows from the definition that for $x \in \mathcal{G}_a$ and $\varphi \in \mathcal{C}(a, b)$ we have $f(x\varphi) = f(x)$.

For $\alpha, \beta \in \mathcal{C}_a$, we define

$$\langle \alpha, \beta \rangle = f(\alpha \times \beta),$$

which defines a (generally indefinite) inner product on $\mathcal{G}_a$. Lemma 4(a) implies that for $x \in \mathcal{G}_a$, $y \in \mathcal{G}_b$ and $\varphi \in \mathcal{C}(a, b)$ the following identity holds:

$$\langle x\varphi, y \rangle = \langle x, y\varphi^* \rangle \tag{1}$$

(which justifies the notation φ^*). Furthermore, Lemma 4(c) implies that if $\varphi \in \mathcal{C}(a, b)$ is a monomorphism, then for $x, y \in \mathcal{C}_a$,

$$\langle x\varphi, y\varphi \rangle = f(x\varphi \times y\varphi) = f((x \times y)\varphi) = f(x \times y) = \langle x, y \rangle. \tag{2}$$

It also follows from the definition and the associativity of the product $\times$ that

$$\langle \alpha \times \beta, \gamma \rangle = f(\alpha \times \beta \times \gamma) = \langle \alpha, \beta \times \gamma \rangle \tag{3}$$

for all α, β, γ in $\mathcal{C}_a$. This extends linearly to the identity

$$\langle x \times y, z \rangle = \langle x, y \times z \rangle \tag{4}$$

for all $x, y, z \in \mathcal{G}_a$.

Let

$$\mathcal{N}_a = \{x \in \mathcal{G}_a : \langle x, y \rangle = 0 \text{ for all } y \in \mathcal{G}_a\},$$

then $\mathcal{N}_a$ is an ideal in the algebra $\mathcal{G}_a$, since if $x \in \mathcal{N}_a$, then by (4), we have for all $y, z \in \mathcal{G}_a$, $\langle x \times y, z\rangle = \langle x, y \times z\rangle = 0$, and hence $x \times y \in \mathcal{N}_a$. So we can form the factor $\mathcal{A}_a = \mathcal{G}_a/\mathcal{N}_a$, which is an associative and commutative algebra with a (possibly indefinite) inner product $\langle ., .\rangle$. The coset $\mathcal{N}_a + \mathrm{id}_a$ is an identity element in $\mathcal{A}_a$, which we denote by $\mathbf{1}_a$.

Lemma 6 *Let $\varphi \in \mathcal{C}(a, b)$.*

(a) *If $x \in \mathcal{N}_a$ then $x\varphi \in \mathcal{N}_b$.*

(b) *If $y \in \mathcal{N}_b$ then $y\varphi^* \in \mathcal{N}_a$.*

(c) *If φ is a monomorphism, then $x\varphi \in \mathcal{N}_b$ implies that $x \in \mathcal{N}_a$.*

Proof. (a) To prove that $x\varphi \in \mathcal{N}_b$, we want to prove that $\langle x\varphi, y\rangle = 0$ for all $y \in \mathcal{G}_b$. By (1), $\langle x\varphi, y\rangle = \langle x, y\varphi^*\rangle$, which is 0 as $x \in \mathcal{N}_a$.

(b) To prove that $y\varphi^* \in \mathcal{N}_a$, we want to prove that $\langle y\varphi^*, x\rangle = 0$ for all $x \in \mathcal{G}_a$. Similarly as before, $\langle y\varphi^*, x\rangle = \langle y, x\varphi\rangle = 0$ as $y \in \mathcal{N}_b$.

(c) Assume that $x\varphi \in \mathcal{N}_b$ for some $x \in \mathcal{G}_a$. Then $\langle x\varphi, y\rangle = 0$ for every $y \in \mathcal{G}_b$, in particular, $\langle x\varphi, z\varphi\rangle = 0$ for every $z \in \mathcal{G}_a$. Then by (2), $\langle x, z\rangle = 0$ for every $z \in \mathcal{G}_a$, and so $x \in \mathcal{N}_a$. □

Corollary 7 (a) *The maps $x \mapsto x\varphi$ and $y \mapsto y\varphi^*$ induce linear maps from $\mathcal{A}_a \to \mathcal{A}_b$ and $\mathcal{A}_b \to \mathcal{A}_a$, respectively.*

(b) *The map $y \mapsto y\varphi^*$ induces an algebra homomorphism.*

(c) *If φ is a monomorphism, then the map $x \mapsto x\varphi$ induces an injective algebra homomorphism.*

We need some simple facts about inner products in direct products.

Lemma 8 *Let a, b_1, b_2 be objects, $\varphi_i \in \mathcal{C}(b_i, a)$, and let (η_1, η_2) be a pullback of (φ_2, φ_2). Let $x_i \in \mathcal{G}_{b_i}$, then*

$$\langle x_1\eta_1^*, x_2\eta_2^*\rangle = \langle x_1\varphi_1, x_2\varphi_2\rangle.$$

In particular if $a = t$, then

$$\langle x_1\eta_1^*, x_2\eta_2^*\rangle = f(x_1)f(x_2),$$

and for $x_i, y_i \in \mathcal{G}_{b_i}$,

$$\langle x_1\eta_1^* \times x_2\eta_2^*, y_1\eta_1^* \times y_2\eta_2^*\rangle = f(x_1 \times y_1)f(x_2 \times y_2).$$

Proof. The first assertion follows from Lemma 5:

$$\begin{aligned}\langle x_1\varphi_1, x_2\varphi_2\rangle &= f(x_1\varphi_1 \times x_2\varphi_2) = f((x_1\eta_1^* \times x_2\eta_2^*)(\varphi_1 \times \varphi_2)) \\ &= f(x_1\eta_1^* \times x_2\eta_2^*) = \langle x_1\eta_1^*, x_2\eta_2^*\rangle.\end{aligned}$$

For the second assertion, it suffices to note that if $a = t$, then by the multiplicativity of f,

$$f(x_1\varphi_1 \times x_2\varphi_2) = f(x_1\varphi_1)f(x_2\varphi_2) = f(x_1)f(x_2),$$

and using that η_i^* is an algebra homomorphism,

$$\begin{aligned}\langle x_1\eta_1^* \times x_2\eta_2^*,\ y_1\eta_1^* \times y_2\eta_2^*\rangle &= f(x_1\eta_1^* \times x_2\eta_2^* \times y_1\eta_1^* \times y_2\eta_2^*) \\ &= f(x_1\eta_1^* \times y_1\eta_1^* \times x_2\eta_2^* \times y_2\eta_2^*) = f((x_1 \times y_1)\eta_1^* \times (x_2 \times y_2)\eta_2^*) \\ &= f(x_1 \times y_1)f(x_2 \times y_2).\end{aligned}$$

□

4 Semidefiniteness

From now on we assume that $\langle .,.\rangle$ is positive semidefinite on every $\mathcal{G}_a$ (and hence positive definite on $\mathcal{A}_a$). This is clearly equivalent with the condition that the matrices $N(f,a)$ are positive semidefinite.

Lemma 9 *The algebra* $\mathcal{A}_a$ *is finite dimensional and* $\dim(\mathcal{A}_a) \le f(a)$.

(The proof, which is an extension of Szegedy's argument in [17], only uses that $N(f, a \times a)$ is positive semidefinite.)

Proof. Let $\pi_1, \pi_2 \in \mathcal{C}(a \times a, a)$ be the canonical projections of $a \times a$ onto a. There is a unique morphism $\varphi \in \mathcal{C}(a, a \times a)$ (the "diagonal embedding") such that $\varphi\pi_1 = \varphi\pi_2 = \mathrm{id}_a$.

Let $e_1, \dots, e_N$ be mutually orthogonal unit vectors in $\mathcal{A}_a$. Both assertions will follow if we prove that $N \le f(a)$.

Let

$$x = \sum_{i=1}^{N}(e_i\pi_1^* \times e_i\pi_2^*) - [\varphi].$$

Then

$$\begin{aligned}\langle x, x\rangle = &\sum_{i=1}^{N}\langle e_i\pi_1^* \times e_i\pi_2^*, e_i\pi_1^* \times e_i\pi_2^*\rangle \\ &+ 2\sum_{i<j}\langle e_i\pi_1^* \times e_i\pi_2^*, e_j\pi_1^* \times e_j\pi_2^*\rangle \\ &- 2\sum_{i=1}^{N}\langle e_i\pi_1^* \times e_i\pi_2^*, \varphi\rangle \\ &+ \langle \varphi, \varphi\rangle. \end{aligned} \tag{5}$$

Here using Lemma 8,

$$\langle e_i\pi_1^* \times e_i\pi_2^*, e_i\pi_1^* \times e_i\pi_2^*\rangle = f(e_i \times e_i)^2 = \langle e_i, e_i\rangle^2 = 1.$$

Similarly,

$$\langle e_i\pi_1^* \times e_i\pi_2^*, e_j\pi_1^* \times e_j\pi_2^*\rangle = \langle e_i, e_j\rangle^2 = 0.$$

Furthermore, using that

$$e_i\pi_2^* \times \varphi = (e_i\pi_2^*\varphi^*)\varphi = (e_i(\varphi\pi_2)^*)\varphi = e_i\varphi,$$

we have

$$\langle e_i\pi_1^* \times e_i\pi_2^*, \varphi\rangle = \langle e_i\pi_1^*, e_i\pi_2^* \times \varphi\rangle = \langle e_i\pi_1^*, e_i\varphi\rangle = \langle e_i, e_i\varphi\pi_1\rangle = \langle e_i, e_i\rangle = 1.$$

Since $[\varphi]$ is an idempotent in $\mathcal{G}_{a\times a}$,

$$\langle \varphi, \varphi\rangle = f(\varphi \times \varphi) = f(\varphi).$$

Hence by (5),

$$\langle x, x\rangle = N + 0 - 2N + f(\varphi) = f(a) - N.$$

Since this is nonnegative, the lemma follows. □

Since $\langle x \times y, z\rangle = \langle x, y \times z\rangle$ for all $x, y, z \in \mathcal{A}_a$, the algebra $\mathcal{A}_a$ has a (unique) orthogonal basis B_a consisting of idempotents. Every idempotent in $\mathcal{A}_a$ is the sum of a subset of B_a, and in particular

$$\mathbf{1}_a = \sum_{p\in B_a} p. \tag{6}$$

Let $\varphi \in \mathcal{C}(a,b)$. Since $\varphi^* \mathcal{A}_b \to \mathcal{A}_a$ is an algebra homomorphism, $p\varphi^*$ is an idempotent in $\mathcal{A}_a$ for any $p \in B_b$, and $\mathbf{1}_b\varphi^* = \mathbf{1}_a$. So (6) implies that

$$\sum_{p \in B_b} p\varphi^* = \mathbf{1}_b\varphi^* = \mathbf{1}_a = \sum_{q \in B_a} q. \tag{7}$$

For $p \in B_b$ and $\varphi \in \mathcal{C}(a,b)$, define

$$B_{\varphi,p} := \{q \in B_a : \ p\varphi^* \times q = q\}.$$

By (7),

$$p\varphi^* = \sum_{q \in B_{\varphi,p}} q. \tag{8}$$

Lemma 10 *Let $p \in B_b$, $q \in B_a$, and $\varphi \in \mathcal{C}(a,b)$.*

(a) *$q \in B_{\varphi,p}$ if and only if*

$$q\varphi = \frac{f(q)}{f(p)}p.$$

(b) *If $q \in B_{\varphi,p}$ and φ is a monomorphism, then $q\varphi = p$.*

Note that here $f(q) = f(q \times q) = \langle q, q\rangle > 0$ and similarly $f(p) > 0$.

Proof. (a) To prove the necessity of the condition, assume that $p' \in B_b \backslash \{p\}$. Then

$$\langle q\varphi, p'\rangle = \langle q, p'\varphi^*\rangle = 0 = \langle \frac{f(q)}{f(p)}p, p'\rangle,$$

since $\langle p, p'\rangle = 0$. Moreover,

$$\langle q\varphi, p\rangle = \langle q, p\varphi^*\rangle = f(q \times (p\varphi^*)) = f(q) = \langle \frac{f(q)}{f(p)}p, p\rangle,$$

since $\langle p, p\rangle = f(p \times p) = f(p)$.

The proof of sufficiency is easy, since q belongs to $B_{\varphi,p'}$ for some $p' \in B_b$, hence $q\varphi = \frac{f(q)}{f(p')}p'$, and so $p = p'$.

(b) Notice that φ defines an algebra homomorphism from $\mathcal{A}_a$ to $\mathcal{A}_b$ by Corollary 7(c), and hence using Lemma 4(c),

$$\frac{f(q)}{f(p)}p = q\varphi = (q \times q)\varphi = (q\varphi) \times (q\varphi) = \Big(\frac{f(q)}{f(p)}p\Big) \times \Big(\frac{f(q)}{f(p)}p\Big) = \Big(\frac{f(q)}{f(p)}\Big)^2 p,$$

which implies that $f(q)/f(p) = 1$. $\square$

5 Simplified idempotents

Let a and b be two objects and $x \in \mathcal{A}_a$, $y \in \mathcal{A}_b$. We say that y is a *simplification* of x if there exists a monomorphism $\varphi \in \mathcal{C}(b,a)$ such that $x = y\varphi$. It is clear that a simplification of a simplification is a simplification.

Lemma 11 *Every $x \in \mathcal{A}_a$ has a unique simplification y such that for every other simplification z of x, y is a simplification of z.*

Proof. Condition C1(b) implies that there is a simplification y of x such that y has no simplification other than itself. We claim that if z is any other simplification of x, then y is a simplification of z.

Let $y \in \mathcal{A}_b$ and $z \in \mathcal{A}_c$, and let $\varphi \in \mathcal{C}(b,a)$ and $\psi \in \mathcal{C}(c,a)$ be monomorphisms such that $x = y\varphi = z\psi$. Then

$$x = y\varphi = (\mathbf{1}_b \times y)\varphi = \mathbf{1}_b\varphi \times y\varphi = \mathbf{1}_b\varphi \times z\psi.$$

By Lemma 5, this implies that, setting $d := p(\varphi,\psi)$, there is a $u \in \mathcal{A}_d$ such that $x = \mathbf{1}_b\varphi \times z\psi = u(\varphi \times \psi)$. Since $\varphi \times \psi$, $\psi^*(\varphi)$ and $\varphi^*(\psi)$ are monomorphisms, this implies that u is a simplification of each of x, y and z. So we must have $u = y$, which implies that y is a simplification of z as claimed. $\square$

So it follows that every $x \in \mathcal{A}_a$ has a "most simplified" version, which we denote by $s(x)$.

Lemma 12 *If p is a basic idempotent, then every simplification of p is a basic idempotent.*

Proof. Let $p \in \mathcal{A}_a$, $y \in \mathcal{A}_b$ and $p = y\varphi$, where $\varphi \in \mathcal{C}(b,a)$ is a monomorphism. Write $y = \sum_{q \in B_b} \lambda_q q$. Then $p = y\varphi = \sum_{q \in B_b} \lambda_q q\varphi$. By Lemma 10, the algebra elements $q\varphi$ are basic idempotents in $\mathcal{A}_a$, and so one of them must be equal to p. Hence $q\varphi = y\varphi$ for this basic idempotent, and by Corollary 7(c), this implies that $y = q$. $\square$

Basic idempotents of the form $s(p)$ will be called *simplified*.

Lemma 13 *Let $\varphi \in \mathcal{C}(b,a)$ be an epimorphism, let $p \in B_a$ be a simplified basic idempotent, and let $q \in B_{\varphi,p}$ Let $s(q) \in B_d$. Then there is an epimorphism $\eta \in \mathcal{C}(d,a)$ such that $s(q) \in B_{\eta,p}$.*

Proof. Let $\mu \in \mathcal{C}(d,b)$ be a monomorphism such that $q = s(q)\mu$. By condition C3, $\mu\varphi$ also factors as $\alpha\beta$, where α is an epimorphism and β is a monomorphism. Then

$$p = \frac{f(p)}{f(q)} q\varphi = \frac{f(p)}{f(q)} s(q)\mu\varphi = \frac{f(p)}{f(q)} s(q)\alpha\beta.$$

Since p is simplified, this implies that $p = \frac{f(p)}{f(q)} s(q)\alpha\sigma$ for some isomorphism σ. Setting $\eta = \alpha\sigma$, we get that $s(q) \in B_{\eta,p}$ by Lemma 10. □

Lemma 14 *If $p \in \mathcal{A}_a$ is a simplified basic idempotent, then for every object b,*

$$\dim \mathcal{A}_b \geq \frac{|\mathcal{C}^{\mathrm{mon}}(a,b)|}{|\mathcal{C}^{\mathrm{mon}}(a,a)|}.$$

Proof. For every $\varphi \in \mathcal{C}^{\mathrm{mon}}(a,b)$, $p\varphi$ is a basic idempotent in $\mathcal{A}_b$. We claim that if $p\varphi = p\psi$, then $[\psi] = [\varphi]$. This will imply that $\mathcal{A}_b$ has at least $|\mathcal{C}^{\mathrm{mon}}(a,b)|/|\mathcal{C}^{\mathrm{mon}}(a,a)|$ different basic idempotents, which will imply the Lemma.

Let $q := p\psi = p\varphi$. Let $\sigma = \varphi \times \psi \in \mathcal{C}(c,b)$. By Lemma 5, there is a $z \in \mathcal{A}_c$ such that $p\varphi \times p\psi = z(\varphi \times \psi)$. But $p\varphi \times p\psi = q \times q = q = p\varphi$, and so $z\varphi^*(\psi)\varphi = z(\varphi \times \psi) = q = p\varphi$, whence $z\varphi^*(\psi) = p$ as φ is monic. But $\varphi^*(\psi)$ is also monic, and since p is simplified, it follows that it is an isomorphism. Similarly, $\psi^*(\varphi)$ is an isomorphism, and hence $\psi = (\psi^*(\varphi))^{-1}\varphi^*(\psi)\varphi$, where $(\psi^*(\varphi))^{-1}\varphi^*(\psi)$ is an automorphism of a. Thus $[\psi] = [\varphi]$. □

Our next goal is to prove that the number of simplified basic idempotents is finite. This is where we also use the existence of a generator object g.

Lemma 15 *For every object g, the following are equivalent.*

(i) *g is a generator.*

(ii) *Every object a has a monomorphism into the direct power $g^{|\mathcal{C}(a,g)|}$.*

(iii) *For any two objects a, b and any two different morphisms $\alpha, \beta \in \mathcal{C}(b,a)$ there is a morphism $\eta \in \mathcal{C}(a,g)$ such that $\alpha\eta \neq \beta\eta$.*

Proof. Clearly (ii) is a sharper form of (i), so it suffices to prove that (i)⇒(iii) and (iii)⇒(ii).

(i)⇒(iii). We know that there is a k such that a has a monomorphism ξ into g^k. Then $\alpha\xi \neq \beta\xi$. Let $\pi_1, \ldots, \pi_k$ be the canonical morphisms of g^k

into g, then by the definition of pullback, there is an $i \in \{1, \dots, k\}$ such that $\alpha\xi\pi_i \neq \beta\xi\pi_i$. So we can take $\eta = \xi\pi_i$.

(iii)⇒(ii). Let $\mathcal{C}(a, g) = \{\varphi_1, \dots, \varphi_k\}$. By the definition of pullbacks, there is a map $\xi \in \mathcal{C}(a, g^k)$ such that $\xi\pi_i = \varphi_i$ for $i \in \{1, \dots, k\}$. We claim that ξ is a monomorphism. Indeed, for any two different morphisms $\alpha, \beta \in \mathcal{C}(b, a)$ there is an i such that $\alpha\varphi_i \neq \beta\varphi_i$, and hence $\alpha\xi \neq \beta\xi$. □

Lemma 16 *The number of simplified basic idempotents is finite.*

Proof. Let a be an object such that $\mathcal{A}_a$ has a simplified basic idempotent p. Let m be the smallest integer such that a has a monomorphism into g^m. By Lemma 15, $|\mathcal{C}(a, g)| \geq m$. Hence it follows that $|\mathcal{C}(a, g^k)| \geq m^k$, and so

$$|\mathcal{C}^{\text{mon}}(a, g^k)| \geq |\mathcal{C}(a, g^{k-m})||\mathcal{C}^{\text{mon}}(a, g^m)| \geq m^{k-m}$$

for $k \geq m$. Combining with Lemma 14, we get that

$$\dim \mathcal{A}_{g^k} \geq \frac{|\mathcal{C}^{\text{mon}}(a, g^k)|}{|\mathcal{C}^{\text{mon}}(a, a)|} \geq \frac{m^{k-m}}{|\mathcal{C}^{\text{mon}}(a, a)|}.$$

Using Lemma 9, we get that

$$m^{k-m} \leq |\mathcal{C}^{\text{mon}}(a, a)| f(g^k) = |\mathcal{C}^{\text{mon}}(a, a)| f(g)^k.$$

Letting $k \to \infty$, we get

$$m \leq f(g).$$

So it follows that a has a monomorphism into $g^{\lfloor f(g) \rfloor}$. By Condition C1, the number of such objects a is finite. □

6 Conclusion

We say that a simplified basic idempotent $p \in \mathcal{A}_a$ is *maximal*, if whenever $\eta \in \mathcal{C}(b, a)$ is an epimorphism and $q \in B_{\eta,p}$ is a simplified basic idempotent, then η is an isomorphism. Lemma 16 implies that there is at least one maximal simplified basic idempotent.

Lemma 17 *Let $\varphi \in \mathcal{C}(b,a)$ be an epimorphism, and let $p \in B_a$ be a maximal simplified basic idempotent. Then*

$$p\varphi^* = \sum_{\substack{\psi\in\mathcal{C}(a,b)\\ \psi\varphi=\mathrm{id}_a}} p\psi.$$

Proof. Let $q \in B_b$. We want to prove that $q \in B_{p,\varphi}$ if and only if $q = p\psi$ for some $\psi \in \mathcal{C}(a,b)$ with $\psi\varphi = \mathrm{id}_a$.

If $q = p\psi$ for such a ψ, then $q\varphi = p\psi\varphi = p$, and so $q \in B_{\varphi,p}$ by Lemma 10.

Conversely, let $q \in B_{\varphi,p}$. By Lemma 13, there is an epimorphism $\eta \in \mathcal{C}(d,a)$ (where $s(q) \in B_d$) such that $s(q) \in B_{\eta,p}$. By the maximality of p, this implies that $d = a$ and $s(q) = p\sigma$ for some isomorphism $\sigma \in \mathcal{C}(a,a)$. It follows that $q = s(q)\mu = p\sigma\mu$ for some monomorphism $\mu \in \mathcal{C}(a,b)$. Let $\beta := \sigma\mu$. Then $p = \frac{f(p)}{f(q)}q\varphi = \frac{f(p)}{f(q)}p\beta\varphi$.

Applying f we see that $f(p) = f(q)$, so $p = q\varphi$. Set $\alpha = \beta\varphi$, so $p = p\alpha$. Write $\alpha = \gamma\delta$, where $\gamma \in \mathcal{C}(a,d)$ is an epimorphism and $\delta \in \mathcal{C}(d,a)$ is a monomorphism. Then $p = p\alpha = (p\gamma)\delta$, and hence by the assumption that p is simplified, it follows that $a = d$ and $\gamma, \delta \in \mathcal{C}(a,a)$ are isomorphisms. Hence α is an isomorphism, and so $\psi = \alpha^{-1}\beta \in \mathcal{C}(a,b)$ is a monomorphism satisfying $\psi\varphi = \mathrm{id}_a$. □

Lemma 18 *For any two objects a, b and maximal simplified basic idempotent $p \in \mathcal{A}_a$,*

$$\sum_{\varphi\in\mathcal{C}(a,b)} p\varphi = f(p)\mathbf{1}_b. \tag{9}$$

Proof. By condition C2(b), the category has a terminal object t. Let $\mathcal{C}(a,t) = \{\alpha\}$ and $\mathcal{C}(b,t) = \{\beta\}$. Set $\gamma = \alpha^*(\beta)$, $\delta = \beta^*(\alpha)$, and $c = p(\alpha,\beta)$.

The algebra $\mathcal{A}_t$ is 1-dimensional, which implies that for any $y \in \mathcal{A}_b$, $y\beta$ is a scalar multiple of $\mathbf{1}_t$, where $f(y\beta) = f(y)$ and the hypothesis that $f(\mathbf{1}_t) = 1$ give the value of the scalar:

$$y\beta = f(y)\mathbf{1}_t. \tag{10}$$

Furthermore, Lemma 4(b) implies that that

$$(y\beta)\alpha^* = (y\delta^*)\gamma. \tag{11}$$

For each $\varphi \in \mathcal{C}(a,b)$, there is a unique $\psi \in \mathcal{C}(a,p(\alpha,\beta))$ with $\psi\gamma = \mathrm{id}_a$ and $\psi\delta = \varphi$. Hence, with Lemma 17,

$$\sum_{\varphi\in\mathcal{C}(a,b)} p\varphi = \sum_{\substack{\psi \\ \alpha\psi=\mathrm{id}_a}} p\psi\delta = \Big(\sum_{\substack{\psi\in\mathcal{C}(a,p(\alpha,\beta)) \\ \psi\gamma=\mathrm{id}_a}} p\psi\Big)\delta = (p\gamma^*)\delta.$$

By (1), (10) and (11), we have for each $y \in \mathcal{A}_b$:

$$\begin{aligned}\langle y, p\gamma^*\delta\rangle &= \langle y\delta^*, p\gamma^*\rangle = \langle y\delta^*\gamma, p\rangle = \langle (y\beta)\alpha^*, p\rangle \\ &= \langle y\beta, p\alpha\rangle = f(y)f(p)\langle \mathbf{1}_t, \mathbf{1}_t\rangle = f(y)f(p) = \langle y, f(p)\mathbf{1}_b\rangle.\end{aligned}$$

This implies that $(p\gamma^*)\delta = f(p)\mathbf{1}_b$. □

We are now ready to prove our main theorem.

Proof of Theorem 1. Let $p \in \mathcal{C}_a$ be a maximal simplified basic idempotent. Then for every object b, by Lemma 18,

$$f(b) = f(\mathbf{1}_b) = \frac{1}{f(p)} \sum_{\varphi\in\mathcal{C}(a,b)} f(p\varphi) = |\mathcal{C}(a,b)|.$$

□

7 Concluding remarks

Homomorphisms between graphs and their number occur in several other contexts. Which of these results can be extended to categories? Let us discuss some examples.

- Questions of existence of homomorphisms between graphs can often be posed in a very clean form using categorial language (see e.g. [8]).
- Counting homomorphisms has been a main tool in proving cancellation laws for finite relational structures [9]. These results were extended to locally finite categories much in the spirit of this paper [10, 14].
- Counting homomorphisms from fixed graphs into a growing sequence of "large" graphs can be used to define convergence of sequences of graphs and their limit objects [5, 12]. Counting homomorphisms from "large"

graphs into fixed graphs (usually with weights) connects this subject to statistical physics. Some of these methods have been extended to hypergraphs and other structures [6]. It would be very interesting to extend these notions and results to categories. One can generalize the notions of cut distance and convergence in a rather straightforward way, but it seems to be much harder to generalize some of the basic proofs, and to find interesting special categories to which the general results would apply.

- The set of homomorphisms between two graphs can be endowed with the structure of a convex cell complex [2], which allows the use of methods from algebraic topology to prove non-existence results concerning homomorphism, in particular colorings [3, 4]. Can this be extended to categories? Again, one can generalize the definitions in more than one way, but the generalization of the results, and even more finding interesting further special cases, is open.

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