I. Matchings and coverings

1. Matchings, covers, and Gallai's theorem

Let $G = (V, E)$ be a graph.[1] A *stable set* is a subset C of V such that $e \not\subseteq C$ for each edge e of G. A *vertex cover* is a subset W of V such that $e \cap W \neq \emptyset$ for each edge e of G. It is not difficult to show that for each $U \subseteq V$:

(1) $\qquad U$ is a stable set $\iff V \setminus U$ is a vertex cover.

A *matching* is a subset M of E such that $e \cap e' = \emptyset$ for all $e, e' \in M$ with $e \neq e'$. A matching is called *perfect* if it covers all vertices (that is, has size $\frac{1}{2}|V|$). An *edge cover* is a subset F of E such that for each vertex v there exists $e \in F$ satisfying $v \in e$. Note that an edge cover can exist only if G has no isolated vertices.

Define:

(2)
$$\begin{aligned}
\alpha(G) &:= \max\{|C| \mid C \text{ is a stable set}\},\\
\tau(G) &:= \min\{|W| \mid W \text{ is a vertex cover}\},\\
\nu(G) &:= \max\{|M| \mid M \text{ is a matching}\},\\
\rho(G) &:= \min\{|F| \mid F \text{ is an edge cover}\}.
\end{aligned}$$

These numbers are called the *stable set number*, the *vertex cover number*, the *matching number*, and the *edge cover number* of G, respectively.

It is not difficult to show that:

(3) $\qquad \alpha(G) \leq \rho(G)$ and $\nu(G) \leq \tau(G)$.

The triangle K_3 shows that strict inequalities are possible. In fact, equality in one of the relations (3) implies equality in the other, as Gallai [6,7] proved:

Theorem 1 (Gallai's theorem)**.** *If $G = (V, E)$ is a graph without isolated vertices, then*

(4) $\qquad \alpha(G) + \tau(G) = |V| = \nu(G) + \rho(G).$

Proof. The first equality follows directly from (1).

To see the second equality, first let M be a matching of size $\nu(G)$. For each of the $|V| - 2|M|$ vertices v missed by M, add to M an edge covering v. [2] We obtain an edge cover F of size $|M| + (|V| - 2|M|) = |V| - |M|$. Hence $\rho(G) \leq |F| = |V| - |M| = |V| - \nu(G)$.

[1] A *graph* is a pair (V, E), where V is a finite set and E is a collection of unordered pairs from V. The elements of V and E are called *vertices* and *edges*, respectively. We denote an edge $\{u, v\}$ sometimes by uv.

[2] M *misses* a vertex u if $u \notin \bigcup M$. Here $\bigcup M$ denotes the union of the edges in M; that is, the set of vertices covered by the edges in M.

Second, let F be an edge cover of size $\rho(G)$. Choose from each component of the graph (V, F) one edge, to obtain a matching M. As (V, F) has at least $|V| - |F|$ components[3], we have $\nu(G) \geq |M| \geq |V| - |F| = |V| - \rho(G)$. ∎

This proof also shows that if we have a matching of maximum cardinality in any graph G, then we can derive from it a minimum cardinality edge cover, and conversely.

2. M-augmenting paths

Basic in matching theory are M-augmenting paths, which are defined as follows. Let M be a matching in a graph $G = (V, E)$. A path $P = (v_0, v_1, \ldots, v_t)$ in G is called *M-augmenting* if

(5) (i) t is odd,
(ii) $v_1v_2, v_3v_4, \ldots, v_{t-2}v_{t-1} \in M$,
(iii) $v_0, v_t \notin \bigcup M$.

Note that this implies that $v_0v_1, v_2v_3, \ldots, v_{t-1}v_t$ do not belong to M.

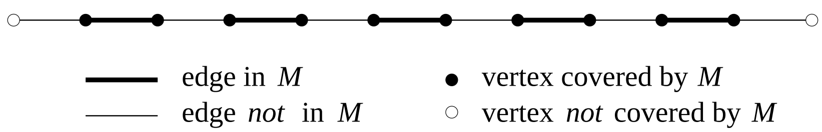


Figure 1

Clearly, if $P = (v_0, v_1, \ldots, v_t)$ is an M-augmenting path, then

(6) $$M' := M \triangle EP$$

is a matching satisfying $|M'| = |M| + 1$.[4]

In fact, it is not difficult to show that:

Theorem 2. *Let $G = (V, E)$ be a graph and let M be a matching in G. Then either M is a matching of maximum cardinality, or there exists an M-augmenting path.*

Proof. If M is a maximum-cardinality matching, there cannot exist an M-augmenting path P, since otherwise $M \triangle EP$ would be a larger matching.

If M' is a matching larger than M, consider the components of the graph $G' := (V, M \cup M')$. Since $|M'| > |M|$, there is components K with more edges in M' than in M. As K has maximum degree at most 2, K is a circuit or a path. As K has more in M' than in M, K forms an M-augmenting path. ∎

[3] *Any* graph (V, E) has at least $|V| - |E|$ components, as can be shown by induction on $|E|$: adding any edge reduces the number of components by at most one.

[4] EP denotes the set of edges in P. $\triangle$ denotes symmetric difference: $X \triangle Y = (X \setminus Y) \cup (Y \setminus X)$.

3. Bipartite matching: Kőnig's theorems

A classical min-max relation due to Kőnig [9] (extending a result of Frobenius [5]) characterizes the maximum size of a matching in a bipartite graph (we follow de proof of De Caen [3]):

Theorem 3 (Kőnig's matching theorem)**.** *For any bipartite graph $G = (V, E)$ one has*

(7) $\qquad \nu(G) = \tau(G).$

That is, the maximum cardinality of a matching in a bipartite graph is equal to the minimum cardinality of a vertex cover.

Proof. By (3) it suffices to show that $\nu(G) \geq \tau(G)$. We may assume that G has at least one edge. Then:

(8) $\qquad G$ has a vertex u covered by each maximum-size matching.

To see this, let $e = uv$ be any edge of G, and suppose that there are maximum-size matchings M and N missing u and v respectively. Let P be the component of $M \cup N$ containing u. So P is a path with end vertex u. Since P is not M-augmenting (as M has maximum size), P has even length, and hence does not traverse v (otherwise, P ends at v, contradicting the bipartiteness of G). So $P \cup e$ would form an N-augmenting path, a contradiction (as N has maximum size). This proves (8).

Now (8) implies that for the graph $G' := G - u$ one has $\nu(G') = \nu(G) - 1$. Moreover, by induction, G' has a vertex cover C of size $\nu(G')$. Then $C \cup \{u\}$ is a vertex cover of G of size $\nu(G') + 1 = \nu(G)$. ∎

Combination of Theorems 1 and 3 yields the following result of Kőnig [10].

Corollary 3a (Kőnig's edge cover theorem)**.** *For any bipartite graph $G = (V, E)$, without isolated vertices, one has*

(9) $\qquad \alpha(G) = \rho(G).$

That is, the maximum cardinality of a stable set in a bipartite graph is equal to the minimum cardinality of an edge cover.

Proof. Directly from Theorems 1 and 3, as $\alpha(G) = |V| - \tau(G) = |V| - \nu(G) = \rho(G)$. ∎

Exercises

3.1. (i) Prove that a k-regular bipartite graph has a perfect matching (if $k \geq 1$).

(ii) Derive that a k-regular bipartite graph has k disjoint perfect matchings.

(iii) Give for each $k > 1$ an example of a k-regular graph not having a perfect matching.

3.2. Prove that in a matrix, the maximum number of nonzero entries with no two in the same line (=row or column), is equal to the minimum number of lines that include all nonzero entries.

3.3. Let $\mathcal{A} = (A_1, \ldots, A_n)$ be a family of subsets of some finite set X. A subset Y of X is called a *transversal* or a *system of distinct representatives* (*SDR*) of $\mathcal{A}$ if there exists a bijection $\pi : \{1, \ldots, n\} \to Y$ such that $\pi(i) \in A_i$ for each $i = 1, \ldots, n$.

Decide if the following collections have an SDR:

(i) $\{3, 4, 5\}, \{2, 5, 6\}, \{1, 2, 5\}, \{1, 2, 3\}, \{1, 3, 6\}$,

(ii) $\{1, 2, 3, 4, 5, 6\}, \{1, 3, 4\}, \{1, 4, 7\}, \{2, 3, 5, 6\}, \{3, 4, 7\}, \{1, 3, 4, 7\}, \{1, 3, 7\}$.

3.4. Let $\mathcal{A} = (A_1, \ldots, A_n)$ be a family of subsets of some finite set X. Prove that $\mathcal{A}$ has an SDR if and only if

$$|\bigcup_{i \in I} A_i| \geq |I| \tag{10}$$

for each subset I of $\{1, \ldots, n\}$.

[Hall's 'marriage' theorem (Hall [8]).]

3.5. Let $\mathcal{A} = (A_1, \ldots, A_n)$ be subsets of the finite set X. A subset Y of X is called a *partial transversal* or a *partial system of distinct representatives* (*partial SDR*) if it is a transversal of some subcollection $(A_{i_1}, \ldots, A_{i_k})$ of $(A_1, \ldots, A_n)$.

Show that the maximum cardinality of a partial SDR of $\mathcal{A}$ is equal to the minimum value of

$$|X \setminus Z| + |\{i \mid A_i \cap Z \neq \emptyset\}|, \tag{11}$$

where Z ranges over all subsets of X.

3.6. Let $\mathcal{A} = (A_1, \ldots, A_n)$ be a family of finite sets and let k be a natural number. Show that $\mathcal{A}$ has k pairwise disjoint SDR's of $\mathcal{A}$ if and only if

$$|\bigcup_{i \in I} A_i| \geq k|I| \tag{12}$$

for each subset I of $\{1, \ldots, n\}$.

3.7. Let $\mathcal{A} = (A_1, \ldots, A_n)$ be a family of subsets of a finite set X and let k be a natural number. Show that X can be partitioned into k partial SDR's if and only if

$$k \cdot |\{i \mid A_i \cap Y \neq \emptyset\}| \geq |Y| \tag{13}$$

for each subset Y of X.

(*Hint:* Replace each A_i by k copies of A_i and use Exercise 3.5 above.)

3.8. Let $(A_1, \ldots, A_n)$ and $(B_1, \ldots, B_n)$ be two partitions of the finite set X.

(i) Show that $(A_1, \ldots, A_n)$ and $(B_1, \ldots, B_n)$ have a *common* SDR if and only if for each subset I of $\{1, \ldots, n\}$, the set $\bigcup_{i \in I} A_i$ intersects at least $|I|$ sets among $B_1, \ldots, B_n$.

(ii) Suppose that $|A_1| = \cdots = |A_n| = |B_1| = \cdots = |B_n|$. Show that the two partitions have a common SDR.

3.9. Let $(A_1,\ldots,A_n)$ and $(B_1,\ldots,B_n)$ be two partitions of the finite set X. Show that the minimum cardinality of a subset of X intersecting each set among $A_1,\ldots,A_n,B_1,\ldots,B_n$ is equal to the maximum number of pairwise disjoint sets in $A_1,\ldots,A_n,B_1,\ldots,B_n$.

3.10. A matrix is called *doubly stochastic* if it is nonnegative and each row sum and each column sum is equal to 1. A matrix is called a *permutation matrix* if each entry is 0 or 1 and each row and each column contains exactly one 1.

(i) Show that for each doubly stochastic matrix $A = (a_{i,j})_{i,j=1}^n$ there exists a permutation $\pi \in S_n$ such that $a_{i,\pi(i)} \neq 0$ for all $i = 1,\ldots,n$.

(ii) Derive that each doubly stochastic matrix is a convex linear combination of permutation matrices.

[Birkhoff-von Neumann theorem (Birkhoff [2], von Neumann [12]).]

3.11. Let $G = (V,E)$ be a bipartite graph and let $b : V \to \mathbb{Z}_+$. Show that G has a subgraph $G' = (V,E')$ such that $\deg_{G'}(v) = b(v)$ for each $v \in V$ if and only if each $X \subseteq V$ contains at least

$$(14) \qquad \frac{1}{2}\Big(\sum_{v\in X} b(v) - \sum_{v\in V\setminus X} b(v)\Big)$$

edges.

3.12. Let $G = (V,E)$ be a bipartite graph and let $k \in \mathbb{N}$. Prove that G has k disjoint perfect matchings if and only if each $X \subseteq V$ contains at least $k(|X| - \frac{1}{2}|V|)$ edges.

3.13. Show that each $2k$-regular graph contains a set F of edges so that each vertex is incident with exactly two edges in F.

4. Cardinality bipartite matching algorithm

We now focus on the problem of finding a maximum-sized matching in a bipartite graph algorithmically.

In any graph, if we have an algorithm finding an M-augmenting path for any matching M (if it exists), then we can find a maximum cardinality matching: we iteratively find matchings $M_0, M_1, \ldots$, with $|M_i| = i$, until we have a matching M_k such that there does not exist any M_k-augmenting path.

We now describe how to find an M-augmenting path in a bipartite graph.

Matching augmenting algorithm for bipartite graphs

input: a bipartite graph $G = (V,E)$ and a matching M,
output: a matching M' satisfying $|M'| > |M|$ (if there is one).
description of the algorithm: Let G have colour classes U and W. Orient each edge $e = \{u,w\}$ of G (with $u \in U, w \in W$) as follows:

$$(15) \qquad \begin{array}{l} \text{if } e \in M \text{ then orient } e \text{ from } w \text{ to } u, \\ \text{if } e \notin M \text{ then orient } e \text{ from } u \text{ to } w. \end{array}$$

Let D be the directed graph thus arising. Consider the sets

(16) $\quad U' := U \setminus \bigcup M$ and $W' := W \setminus \bigcup M$.

Now an M-augmenting path (if it exists) can be found by finding a directed path in D from any vertex in U' to any vertex in W'. Hence in this way we can find a matching larger than M. ∎

This implies:

Theorem 4. *A maximum-size matching in a bipartite graph can be found in time* $O(|V||E|)$.

Proof. The correctness of the algorithm is immediate. Since a directed path can be found in time $O(|E|)$, we can find an augmenting path in time $O(|E|)$. Hence a maximum cardinality matching in a bipartite graph can be found in time $O(|V||E|)$ (as we do at most $|V|$ iterations). ∎

Exercises

4.1. Find a maximum-size matching and a minimum vertex cover in the bipartite graph in Figure 2.

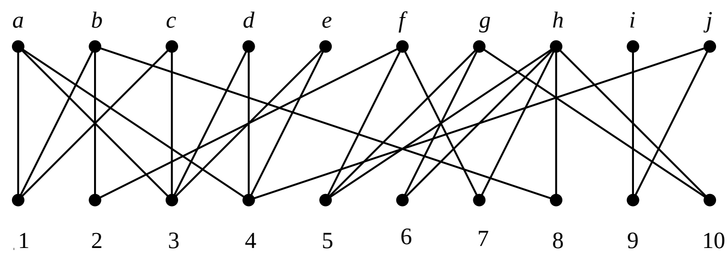


Figure 2

4.2. Derive Kőnig's matching theorem from the cardinality matching algorithm for bipartite graphs.

4.3. Show that a minimum-size vertex cover in a bipartite graph can be found in polynomial time.

4.4. Show that, given a family of sets, a system of distinct representatives can be found in polynomial time (if it exists).

5. Nonbipartite matching: Tutte's 1-factor theorem and the Tutte-Berge formula

A basic result on matchings in arbitrary (not necessarily bipartite) graphs was found by Tutte [13]. It characterizes graphs that have a perfect matching. A *perfect matching* (or a $1-$*factor*) is a matching M that covers all vertices of the graph. (So M partitions the vertex set of G.)

Berge [1] observed that Tutte's theorem implies a min-max formula for the maximum size of a matching in a graph, the Tutte-Berge formula, which we prove first.

Call a component of a graph *odd* if it has an odd number of vertices. For any graph G, define

(17) $$o(G) := \text{number of odd components of } G.$$

Let $\nu(G)$ denotes the maximum size of a matching. For any graph $G = (V, E)$ and $U \subseteq V$, the graph obtained by deleting all vertices in U and all edges incident with U, is denoted by $G - U$.

Then:

Theorem 5 (Tutte-Berge formula)**.** *For each graph $G = (V, E)$,*

(18) $$\nu(G) = \min_{U \subseteq V} \tfrac{1}{2}(|V| + |U| - o(G - U)).$$

Proof. To see $\leq$, we have for each $U \subseteq V$:

(19) $$\nu(G) \leq |U| + \nu(G - U) \leq |U| + \tfrac{1}{2}(|V \setminus U| - o(G - U)) = \tfrac{1}{2}(|V| + |U| - o(G - U)).$$

We prove the reverse inequality by induction on $|V|$, the case $V = \emptyset$ being trivial.

First assume that there exists a vertex v covered by all maximum-size matchings. Then $\nu(G - v) = \nu(G) - 1$, and by induction there exists a subset U' of $V \setminus \{v\}$ with

(20) $$\nu(G - v) = \tfrac{1}{2}(|V \setminus \{v\}| + |U'| - o(G - v - U')).$$

Then $U := U' \cup \{v\}$ gives equality in (18), since

(21) $$\begin{aligned}&\nu(G) = \nu(G - v) + 1 = \tfrac{1}{2}(|V \setminus \{v\}| + |U'| - o(G - v - U')) + 1\\ &= \tfrac{1}{2}(|V| + |U| - o(G - U)).\end{aligned}$$

So we can assume that there is no such v. We show that

(22) there is a matching containing at least $\lfloor \frac{1}{2} K \rfloor$ edges in each component K of G.

This implies $\nu(G) \geq \frac{1}{2}(|V| - o(G))$, and hence we can take $U = \emptyset$ in (18).

To prove (22), suppose to the contrary that for each maximum-size matching M there is a component in which M misses at least two distinct vertices u and v. Among all such M, u, v, choose them such that the distance $\text{dist}(u, v)$ of u and v in G is as small as possible.

If $\text{dist}(u, v) = 1$, then u and v are adjacent, and hence we can augment M by the edge uv, contradicting the maximality of $|M|$. So $\text{dist}(u, v) \geq 2$, and hence we can choose an intermediate vertex t on a shortest $u - v$ path. By assumption, there exists a maximum-size matching N missing t. By the minimality of $\text{dist}(u, v)$, N covers both u and v, and M covers t.

Let P_u, P_t, P_v the components of the graph $(V, M \cup N)$ containing u, t, v, respectively. Each of these is a path containing as many edges in M as in N (otherwise it would be M- or N-augmenting). So $P_u \neq P_v$. Hence, $P_u \neq P_t$ or $P_v \neq P_t$. By symmetry we can assume that $P_u \neq P_t$. Then $N \triangle EP_u$ is a maximum-size matching missing u and t, contradicting our assumption, as $\mathrm{dist}(u,t) < \mathrm{dist}(u,v)$. This proves (22). ▮

(This proof is based on the proof of Lovász [11] of Edmonds' matching polytope theorem.)

The Tutte-Berge formula immediately implies Tutte's 1-factor theorem.

Corollary 5a (Tutte's 1-factor theorem)**.** *A graph $G = (V, E)$ has a perfect matching if and only if $G - U$ has at most $|U|$ odd components, for each $U \subseteq V$.*

Proof. Directly from the Tutte-Berge formula (Theorem 5), since G has a perfect matching if and only if $\nu(G) \geq \frac{1}{2}|V|$. ▮

In the following sections we will show how to find a maximum-size matching algorithmically.

With Gallai's theorem, the Tutte-Berge formula implies a formula for the edge cover number $\rho(G)$, where $o(U)$ denotes the number of odd components of the subgraph $G[U]$ of G induced by U (meaning that $G[U] = (U, \{e \in E \mid e \subseteq U\})$):

Corollary 5b. *Let $G = (V, E)$ be a graph without isolated vertices. Then*

(23) $$\rho(G) = \max_{U \subseteq V} \frac{|U| + o(U)}{2}.$$

Proof. By Gallai's theorem (Theorem 1) and the Tutte-Berge formula (Theorem 5),

(24) $$\rho(G) = |V| - \nu(G) = |V| - \min_{W \subseteq V} \frac{|V| + |W| - o(V \setminus W)}{2} = \max_{U \subseteq V} \frac{|U| + o(U)}{2}.$$ ▮

Exercises

5.1. (i) Show that a tree has at most one perfect matching.

(ii) Show (not using Tutte's 1-factor theorem) that a tree $G = (V, E)$ has a perfect matching if and only if the subgraph $G - v$ has exactly one odd component, for each $v \in V$.

5.2. Let G be a 3-regular graph without any bridge. Show that G has a perfect matching. (A *bridge* is an edge e not contained in any circuit; equivalently, deleting e increases the number of components; equivalently, $\{e\}$ is a cut.)

5.3. Let $G = (V, E)$ be a graph and let T be a subset of V. Then G has a matching covering T if and only if the number of odd components of $G - W$ contained in T is at most $|W|$, for each $W \subseteq V$.

5.4. Let $G = (V, E)$ be a graph and let $b : V \to \mathbb{Z}_+$. Show that G has a subgraph $G' = (V, E')$ such that $\deg_{G'}(v) = b(v)$ for each $v \in V$ if and only if for each two disjoint subsets U and W

of V one has

(25) $$\sum_{v\in U} b(v) \geq q(U,W) + \sum_{v\in W}(b(v) - d_{G-U}(v)).$$

Here $q(U,W)$ denotes the number of components K of $G-(U\cup W)$ for which $b(K)$ plus the number of edges connecting K and W, is odd. Moreover, $d_{G-U}(v)$ is the degree of v in the subgraph induced by $V\setminus U$.

6. Cardinality matching algorithm

We now investigate the problem of finding a maximum-cardinality matching algorithmically. Like in the bipartite case, the key is to find an augmenting path. However, the idea for bipartite graphs to orient the edges using the two colour classes, does not apply to nonbipartite graphs.

Yet one could try to find an M-augmenting path by finding a so-called M-alternating walk, but such a path can run into a loop that cannot immediately be deleted. It was J. Edmonds who found the trick to resolve this problem, namely by 'shrinking' the loop (which he called a 'blossom'). Then applying recursion to a smaller graph solves the problem.

We first describe the operation of *shrinking*. Let X and Y be sets. Then we define X/Y as follows:

(26) $$X/Y := X \text{ if } X\cap Y = \emptyset,$$
$$X/Y := (X\setminus Y)\cup\{Y\} \text{ if } X\cap Y \neq \emptyset.$$

So if $G=(V,E)$ is a graph and $C\subseteq V$, then V/C arises from V by deleting all vertices in C, and adding one new vertex called C. For any edge e of G, $e/C = e$ if e is disjoint from C, while $e/C = uC$ if $e = uv$ with $u\notin C$, $v\in C$. (If $e=uv$ with $u,v\in C$, then e/C is a loop CC; they can be neglected in the context of matchings.) Then for any $F\subseteq E$:

(27) $$F/C := \{f/C \mid f\in F\}.$$

So $G/C := (V/C, E/C)$ is again a graph. We say that G/C arises from G by *shrinking* C.

Let $G=(V,E)$ be a graph and let M be a matching in G, and let W be the set of vertices missed by M. A walk $P=(v_0,v_1,\ldots,v_t)$ is called *M-alternating* if for each $i=1,\ldots,t-1$ exactly one of $v_{i-1}v_i$ and v_iv_{i+1} belongs to M. Note that one can find a shortest M-alternating $W-W$ walk of positive length, by considering the auxiliary directed graph $D=(V,A)$ with

(28) $$A := \{(w,w') \mid \exists x\in V : \{w,x\}\in E, \{x,w'\}\in M\}.$$

Then M-alternating $W-W$ walks correspond to directed walks in D from a vertex in W to a vertex that is adjacent to at least one vertex in W.

So an M-augmenting path is an M-alternating $W-W$ walk of positive length, in which all vertices are distinct. By Theorem 2, a matching M has maximum size if and only if there is no M-augmenting path. We call an M-alternating walk P an *M-blossom* if $v_0, \ldots, v_{t-1}$ are distinct, v_0 is missed by M, and $v_t = v_0$.

The core of the algorithm is the following observation.

Theorem 6. *Let C be an M-blossom in G. Then M has maximum size in G if and only if M/C has maximum size in G/C.*

Proof. Let $C = (v_0, v_1, \ldots, v_t)$, $G' := G/C$ and $M' := M/C$.

First let P be an M-augmenting path in G. We may assume that P does not start at v_0 (otherwise we can reverse P). If P does not traverse any vertex in C, then P is also M'-augmenting in G'. If P does traverse a vertex in C, we can decompose P as $P = QR$, where Q ends at a vertex in C, and no other vertex on Q belongs to C. Then by replacing the last vertex of Q by C makes Q to an M'-augmenting path in G'.

Conversely, let P' be an M'-augmenting path in G'. If P' does not traverse vertex C of G', then P' is also an M-augmenting path in G. If P' traverses vertex C of G', we may assume it ends at C (as C is missed by M'). So we can replace C in P' by some vertex $v_i \in C$ to obtain a path Q in G ending at v_i. If i is odd, extending Q by $v_{i+1}, \ldots, v_{t-1}, v_t$ gives an M-augmenting path in G. If i is even, extending Q by $v_{i-1}, \ldots, v_1, v_0$ gives an M-augmenting path in G. ∎

Another useful observation is (where a *$W - v$ walk* is a walk starting at a vertex in W and ending at v):

Theorem 7. *Let $P = (v_0, v_1, \ldots, v_t)$ be a shortest even-length M-alternating $W - v$ walk. Then either P is simple or there exist $i < j$ such that $v_i = v_j$, i is even, j is odd, and $v_0, \ldots, v_{j-1}$ are all distinct.*

Proof. Assume P is not simple. Choose $i < j$ such that $v_j = v_i$ and such that j is as small as possible. If $j - i$ is even, we can delete $v_{i+1}, \ldots, v_j$ from P so as to obtain a shorter M-alternating $W - v$ walk. So $j - i$ is odd. If j is even and i is odd, then $v_{i+1} = v_{j-1}$ (as it is the vertex matched to $v_i = v_j$), contradicting the minimality of j. ∎

We now describe an algorithm for the following problem:

(29) given: a matching M;
find: a matching N with $|N| = |M| + 1$ or conclude that M is a maximum-size matching.

Let W be the set of vertices missed by M.

(30) **Case 1.** *There is no M-alternating $W - W$ walk.* Then M has maximum size (as there is no M-augmenting path).

Case 2. *There is an M-alternating $W - W$ walk.* Let $P = (v_0, v_1, \ldots, v_t)$ be a shortest such walk.

Case 2a. P is path. Hence P is an M-augmenting path. Then output $N := M \triangle EP$.

Case 2b. P is not a path. That is, not all vertices in P are different. Choose $i < j$ such that $v_i = v_j$ with j as small as possible. Reset $M := M \triangle \{v_0v_1, v_1v_2, \ldots, v_{i-1}v_i\}$. Then $C := (v_i, v_{i+1}, \ldots, v_j)$ is an M-blossom. Apply the algorithm (recursively) to $G' = G/C$ and $M' := M/C$.

- If it gives an M'-augmenting path P' in G', transform P' to an M-augmenting path in G (as in the proof of Theorem 6).
- If it concludes that M' has maximum size in G', then M has maximum size in G (by Theorem 6).

This gives a polynomial-time algorithm to find a maximum-size matching, which is a basic result of Edmonds [4].

Theorem 8. *Given an undirected graph, a maximum-size matching can be found in time* $O(|V|^2|E|)$.

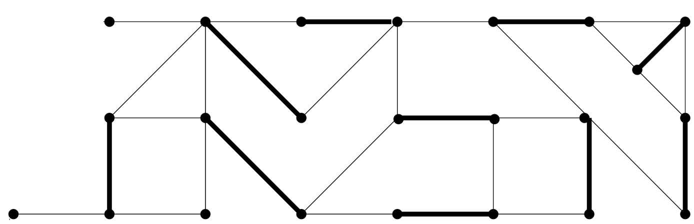
Proof. The algorithm directly follows from algorithm (30), since one can iteratively apply it, starting with $M = \emptyset$, until a maximum-size matching is attained.

By using (28), a shortest M-alternating $W - W$ walk can be found in time $O(|E|)$. Moreover, the graph G/C can be constructed in time $O(|E|)$. Since the recursion has depth at most $|V|$, each application of algorithm (30) takes $O(|V||E|)$ time. Since the number of applications is at most $|V|$, we have the time bound given in the theorem. ∎

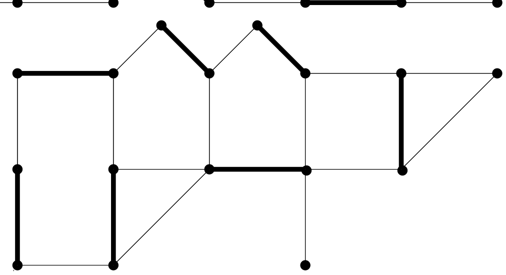
Exercises

6.1. Apply the matching augmenting algorithm to the matchings in the following graphs:

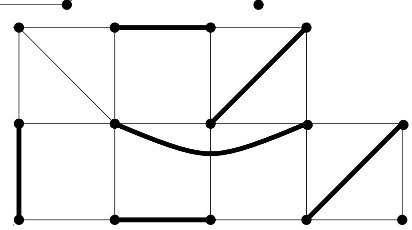
(i)



(ii)



(iii)



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