Composing Models

Jan van Eijck, Floor Sietsma, Yanjing Wang

LOFT 2010, 7 July 2010

Abstract

- We study a new composition operation on (epistemic) multiagent models and update actions that takes vocabulary extensions into account.
- This operation allows to represent partial observational information about a large model in a small model, where the small models can be viewed as representations of the observational power of agents, and about their powers for changing the facts of the world.
- Our investigation provides ways to check relevant epistemic properties on small components of large models, and our approach generalizes the use of 'locally generated models'.

Overview: Three Simple Messages

- Models can be made small by vocabulary restriction
- Composing restricted models is easy
- Compositions of restricted models are useful

Note: an expanded version of this LOFT paper can be found in Chapter 5 of the PhD Thesis of Yanjing Wang, Epistemic Modelling and Protocol Dynamics, to be defended in September 2010 (available upon request from the author).

Multi-agent Models with Different Vocabularies

Fix a set of proposition letters P. Call a subset of P a vocabulary.

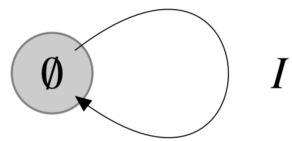
Consider multi-agent models with vocabularies Q taken from P.

Call such models restricted models.

This allows us to refine ‘knowledge about the world’ to ‘knowledge about Q’.

Knowing Nothing About Anything

The restricted model $\mathcal{E}$ for knowing nothing about anything:

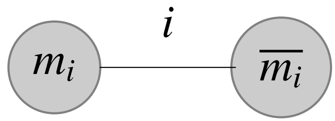


Formally, $(\{e\}, I, \{\{(e,e)\} \mid i \in I\}, e \mapsto \emptyset, \emptyset)$.

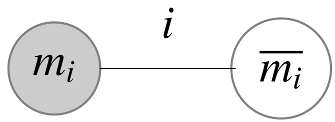
Compare: the non-restricted model for knowing nothing about anything, for a language over P with $|P| = n$ has 2^n worlds.

Restricted Models for Muddy Children

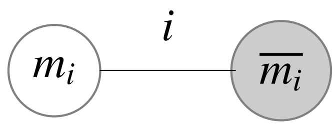
Single child not knowing whether it is muddy. Voc restricted to m_i:



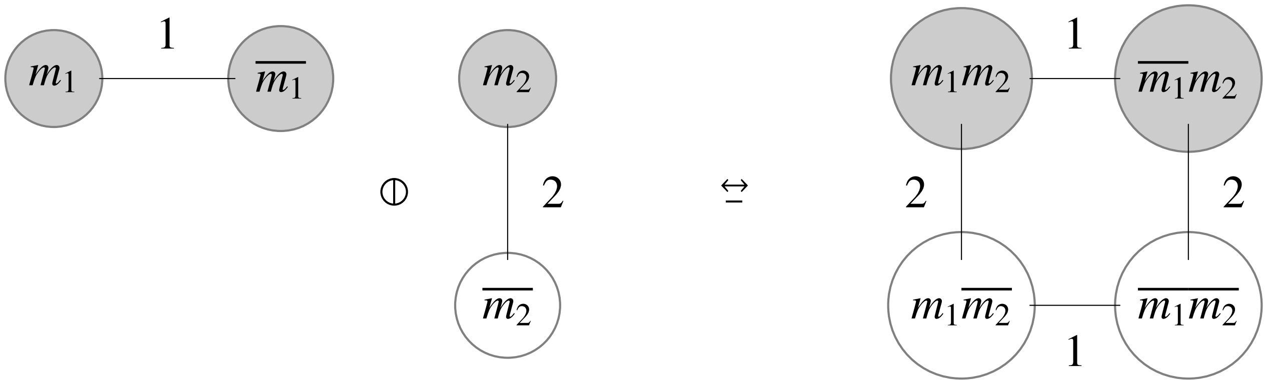
Single muddy child not knowing whether it is muddy:



Single clean child not knowing whether it is muddy:



Restricted Model Composition: Example



Restricted Model Composition: Definition

Restricted model composition is a product construction.

The composition $\mathcal{M} \oplus \mathcal{N}$ of two restricted multi-agent models with the same agent set I is given by $(W, I, R, V, Q_M \cup Q_N)$, where the new set of worlds is given by:

$$W = \{(w, v) \mid w \in W_M, v \in W_N, V_M(w) \cap Q_N = V_N(v) \cap Q_M\},$$

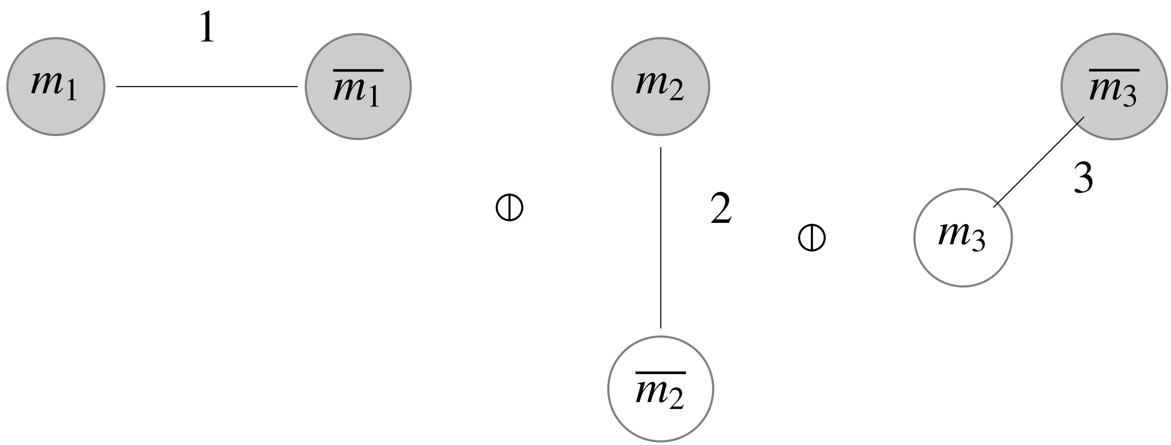
the new accessibility relations are defined as the product of the relations on the components, in the usual product way:

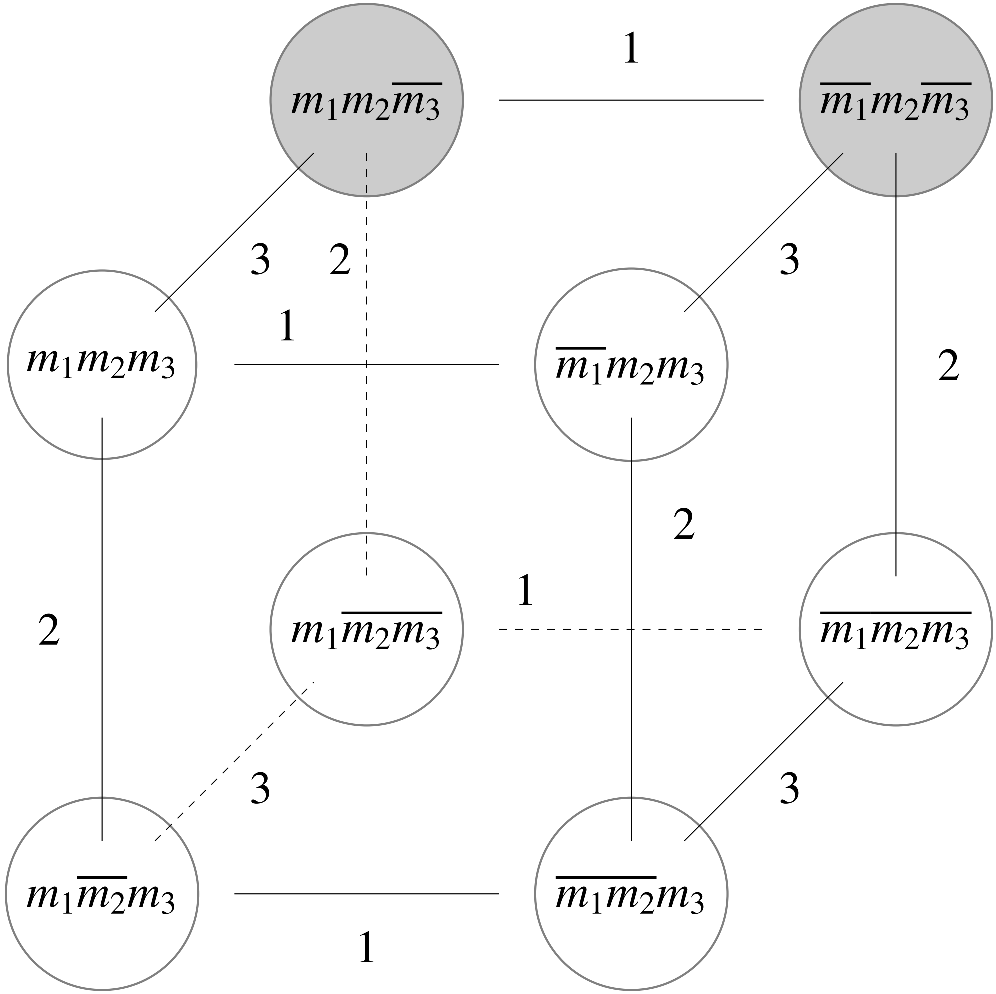
$$(w, v)R_i(w', v') \text{ iff } wR_{iM}w' \text{ and } vR_{iN}v',$$

and $V(w, v)$ agrees with $V_M(w)$ on Q_M and with $V_N(v)$ on Q_N:

$$V(w, v) = V_M(w) \cup V_N(v).$$

Composing the Model for Three Muddy Children




$m_1 m_2 \overline{m_3}$
1
$\overline{m_1} m_2 \overline{m_3}$
3
2
3
$m_1 m_2 m_3$
1
$\overline{m_1} m_2 m_3$
2
2
2
$m_1 \overline{m_2 m_3}$
1
$\overline{m_1 m_2 m_3}$
3
3
$m_1 \overline{m_2} m_3$
$\overline{m_1 m_2} m_3$
1

Structural Properties of $\oplus$

$\underline{\leftrightarrow}$ is a congruence for $\oplus$:

If $\mathcal{M}_1 \underline{\leftrightarrow} \mathcal{M}_2$ and $\mathcal{N}_1 \underline{\leftrightarrow} \mathcal{N}_2$ then $\mathcal{M}_1 \oplus \mathcal{N}_1 \underline{\leftrightarrow} \mathcal{M}_2 \oplus \mathcal{N}_2$.

Multi-agent models form a commutative monoid under $\oplus$:

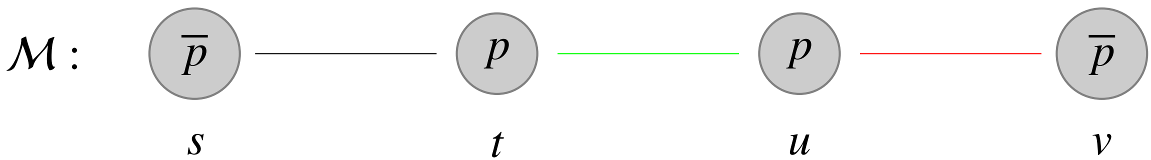
$$\begin{aligned} \mathcal{E} \oplus \mathcal{M} &\underline{\leftrightarrow} \mathcal{M} \\ \mathcal{M} \oplus \mathcal{E} &\underline{\leftrightarrow} \mathcal{M} \\ \mathcal{M} \oplus (\mathcal{N} \oplus \mathcal{K}) &\underline{\leftrightarrow} (\mathcal{M} \oplus \mathcal{N}) \oplus \mathcal{K} \\ \mathcal{M} \oplus \mathcal{N} &\underline{\leftrightarrow} \mathcal{N} \oplus \mathcal{M} \end{aligned}$$

This yields the well-known preordering $\leq$:

$\mathcal{M} \leq \mathcal{N}$ iff there is a $\mathcal{K}$ with $\mathcal{M} \oplus \mathcal{K} \underline{\leftrightarrow} \mathcal{N}$.

$\oplus$ is not idempotent

There are $\mathcal{M}$ with the property that $\mathcal{M} \oplus \mathcal{M} \not\leftrightarrows \mathcal{M}$. Example:



(t, u) is a p-world in $\mathcal{M} \oplus \mathcal{M}$, but (t, u) cannot reach a $\overline{p}$ world in $\mathcal{M} \oplus \mathcal{M}$.

Left-Simulation

A left-simulation between $\mathcal{M}$ and $\mathcal{N}$ is like a bisimulation, but with the **invariance** condition restricted to the vocabulary of $\mathcal{M}$, and with the **zig** condition omitted.

Formally, a left-simulation between $\mathcal{M}$ and $\mathcal{N}$ is a relation $C \subseteq W_M \times W_N$ such that wCv implies that the following hold:

Restricted invariance $p \in V_M(w)$ iff $p \in V_N(v)$ for all $p \in Q_M$,

Zag If for some $i \in I$ there is a $v' \in W_N$ with $v \xrightarrow{i} v'$ then there is a $w' \in W_M$ with $w \xrightarrow{i} w'$ and $w'Cv'$.

$\mathcal{M}, w \leftleftharpoons \mathcal{N}, v$: there is a left-simulation that connects w and v.

$\mathcal{M} \leftleftharpoons \mathcal{N}$: there is a total left-simulation between $\mathcal{M}$ and $\mathcal{N}$

Theorem 1 *If $\mathcal{M} \leq \mathcal{N}$ then $\mathcal{M} \leftleftharpoons \mathcal{N}$.*

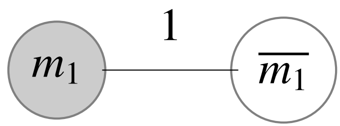
$\mathcal{M}$ is propositionally differentiated if it holds for all worlds w, w' of $\mathcal{M}$ that if w and w' have the same valuation then $w \underline{\leftrightarrow} w'$.

In other words, if $w \not\underline{\leftrightarrow} w'$ then this difference shows up as a difference in the valuations of w and w'.

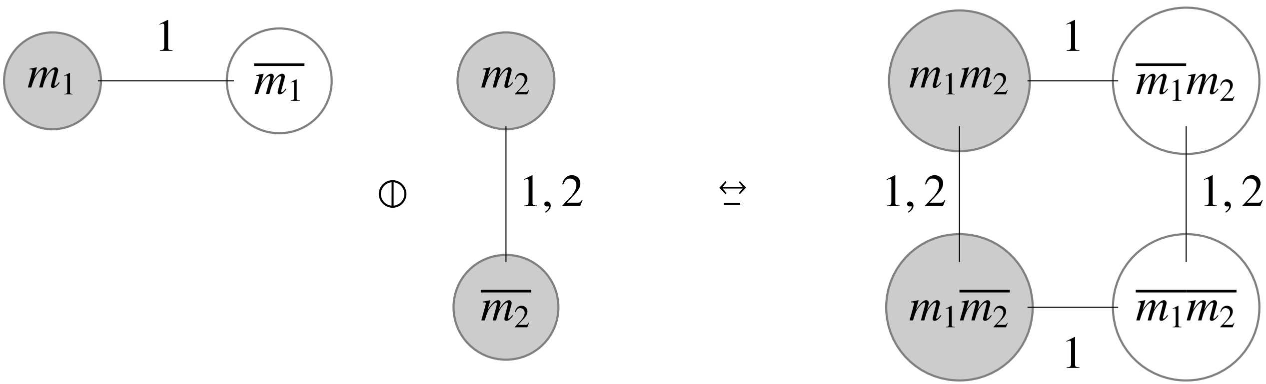
Theorem 2 *If $\mathcal{M}$ is propositionally differentiated, then $\mathcal{M} \underline{\leftarrow} \mathcal{N}$ implies $\mathcal{M} \leq \mathcal{N}$.*

The full paper has an example showing that the theorem may fail for models that are not propositionally differentiated.

Expansion to Larger Vocabulary



Expansion of this model to m_1, m_2:



Vocabulary Expansion, Formally

Let $\mathcal{Q}^I$ be the universal ignorance model for Q, i.e. $\mathcal{Q}^I = (W, I, R, V, Q)$ with $W = \mathcal{P}(Q)$, $R_i = W^2$, $V = \text{id}$.

If $\mathcal{M} = (W, I, R, V, Q)$ is a restricted static model and Q_1 is a set of proposition letters, then we define the expanded model for the larger vocabulary $Q \cup Q_1$ as follows:

$\mathcal{M} \triangleleft Q_1 = \mathcal{M} \oplus \mathcal{Q}_1{}^I$.

Theorem 3 (Preservation) *If a pointed model* $(\mathcal{M}, s)$ *is decomposable into models*

$$(\mathcal{M}_0, s_0), \ldots, (\mathcal{M}_n, s_n)$$

with disjoint vocabularies

$$\mathcal{Q}_0, \mathcal{Q}_1, \ldots, \mathcal{Q}_n,$$

then for any i*:*

$$\mathcal{M}_i, s_i \leftrightarroweq_{\mathcal{Q}_i} \mathcal{M}, s.$$

Therefore for any ϕ *in* $PDL_{\mathcal{Q}_i,\mathsf{Ag}}$*:*

$$\mathcal{M}_i, s_i \models \phi \iff \mathcal{M}, s \models \phi.$$

This means that any properties of the large model that can be stated in a local vocabulary can be checked locally.

Locally Generated Models

We say $\mathcal{M}$ is *locally generated* if, for every agent i, there is a set of boolean formulas Φ_i (the set of local observables) based on $Q_{\mathcal{M}}$ such that for all $w, w' \in W_M$:

$$w \sim_i w' \text{ iff for all } \varphi \in \Phi_i,\ \mathcal{M} \models_w \varphi \Leftrightarrow \mathcal{M} \models_{w'} \varphi$$

Intuitively, a model is locally generated if the local observables of the agents determine the epistemic relations in the model.

Example: the n-Muddy Children model is locally generated by set of observables $\Phi_1, \ldots, \Phi_n$, where

$$\Phi_i = \{m_j \mid j \in I, j \neq i\}.$$

Theorem 4 (Decomposition by agents) *Let a set of agents*

$$Ag = \{1, 2, \ldots n\}$$

be given.

If $\mathcal{M} = (W, Q, Ag, \sim, V)$ *is locally generated by* $\Phi_1, \ldots, \Phi_n$, *then there are models* $\mathcal{M}_1, \ldots, \mathcal{M}_n$ *and* $\mathcal{M}_0$ *such that:*

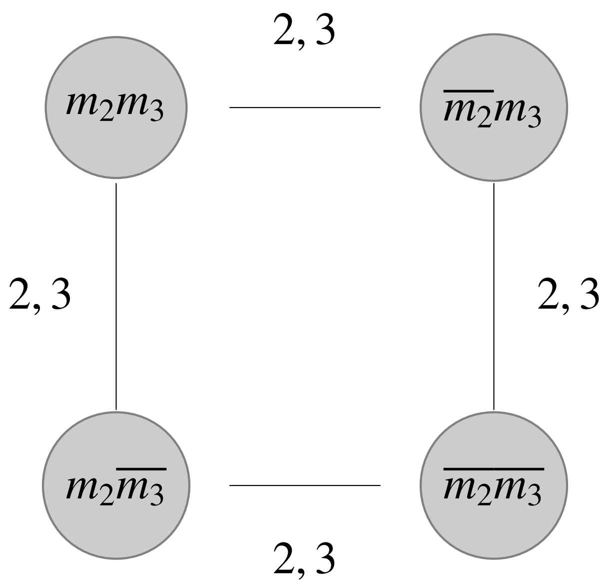
- $\mathcal{M} \leftrightarroweq (\mathcal{M}_0 \oplus \mathcal{M}_1 \oplus \cdots \oplus \mathcal{M}_n)$;
- $|W_{\mathcal{M}_j}| \leq |W|$ *and* $\mathcal{M}_i$ *is a bisimulation contracted model;*
- $Q_{\mathcal{M}_j} = \{p \in Q_{\mathcal{M}} \mid p \textit{ appears in } \Phi_j\}$ *for* $j > 0$.

Another possible decomposition of locally generated models is **by issues**. Example: Our earlier Muddy Children decomposition. See Yanjing's thesis.

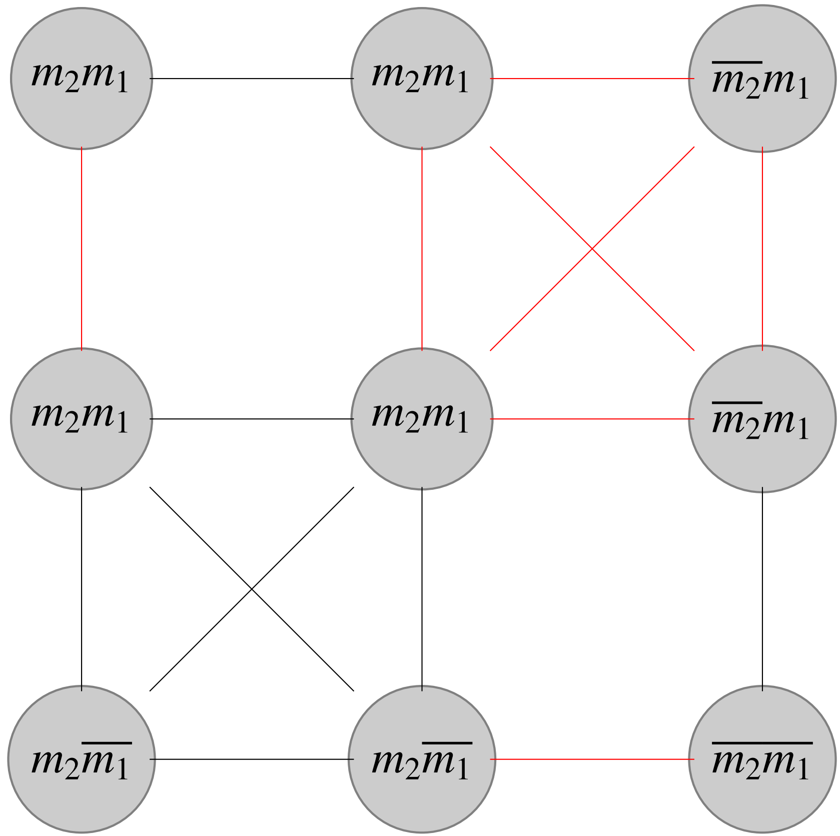
Decomposition by agents of the 3-Muddy Children model, for first agent:

$$\Phi_1 = \{m_2, m_3\}.$$

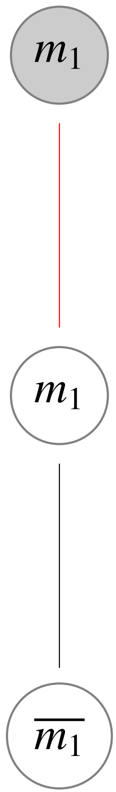
The model $\mathcal{M}_1$ looks like this:



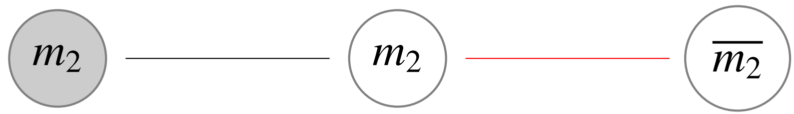
Not locally generated, but decomposable:



Decomposition:

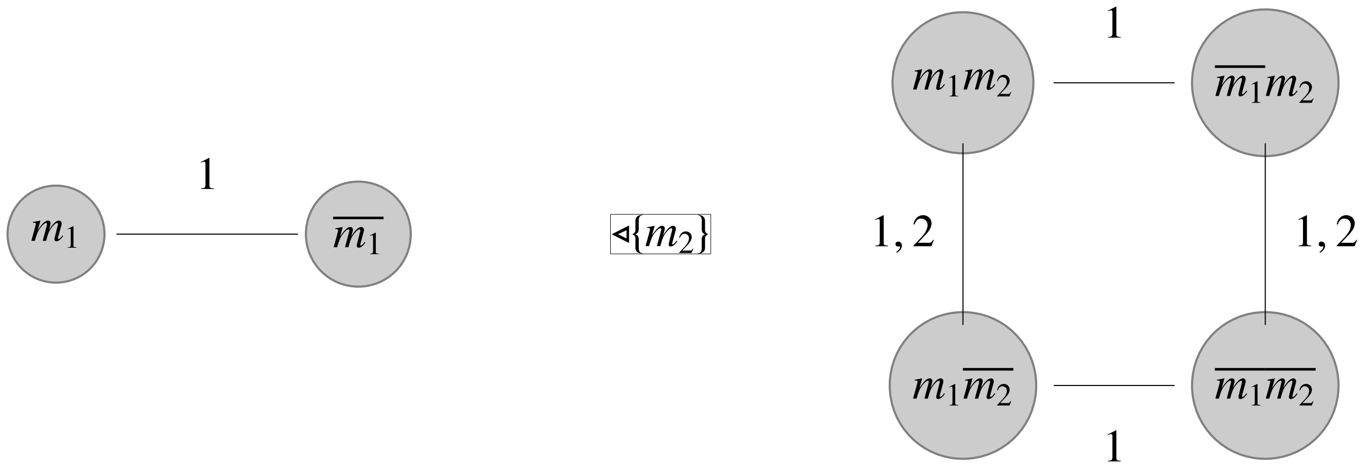


⦶

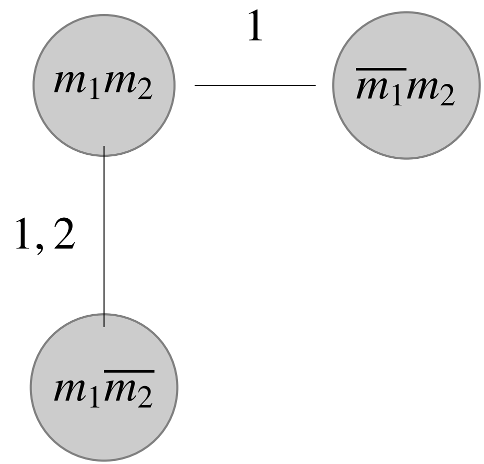


Two muddy children who each have a mirror, but they do not know that of each other.

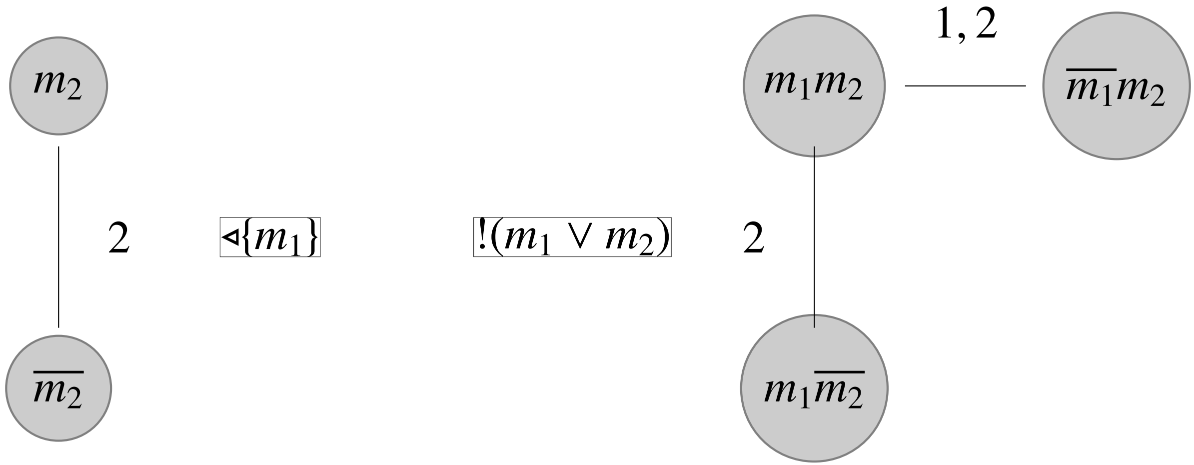
Update with Vocabulary Expansion: Public Announcement

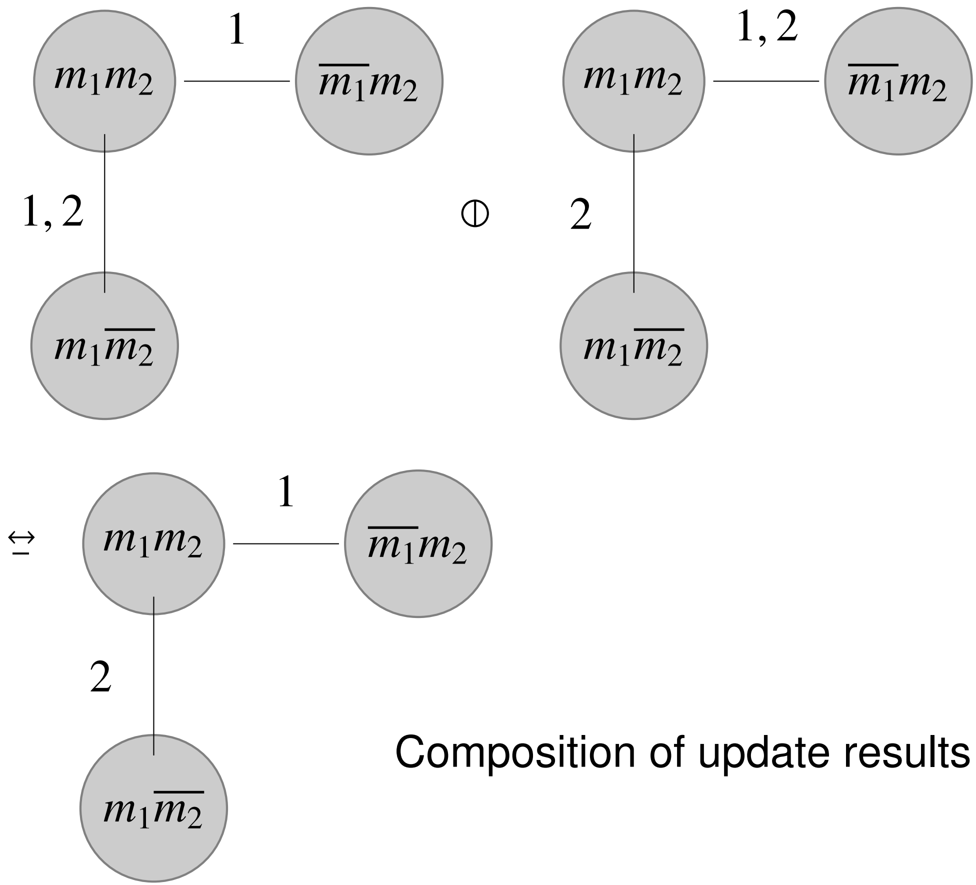


$!(m_1 \vee m_2)$



Update of Other Component





Composition of update results

Interaction of $\oplus$ and $\otimes$

Theorem 5 *If A is propositionally differentiated then:*

$$(\mathcal{M} \oplus \mathcal{N}) \otimes A \leftrightarroweq (\mathcal{M} \otimes A) \oplus (\mathcal{N} \otimes A).$$

And without conditions on the action models, with the appropriate notion of $\oplus$ for action models:

Theorem 6 $\mathcal{M} \otimes (A \oplus B) \leftrightarroweq (\mathcal{M} \otimes A) \oplus (\mathcal{M} \otimes B)$.

Further Work

- Extend DEMO with $\obar$, in order to allow epistemic model checking of large models on local components.
- Characterize models in terms of their composition. (Example: what do models that are composed from only two-world components look like? Answered in the full paper.)
- Study the combination of communicative actions and vocabulary expansion. Example task: axiomatize the strong Kleene logic of public announcement $!\phi$ and vocabulary expansion $\sharp p$, where $\sharp p$ is interpreted as the model changing operation $\mathcal{M} \mapsto \mathcal{M} \triangleleft \{p\}$.
- Work out obvious connections with awareness logics, and with work on the dynamics of awareness.