On the structure of complete sets

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Abstract

The many types of resource bounded reductions that are both object of study and research tool in structural complexity theory have given rise to a large variety of completeness notions. A complete set in a complexity class is a manageable object that represents the structure of the entire class. The study of its structure can reveal properties that are general in that complexity class, and the study of the structure of complete sets in different classes can reveal secrets about the relation between these classes. The research into all sorts of aspects and properties of complete sets has been and will be a major topic in structural complexity theory. In this expository paper we review the progress that has been made in recent years on selected topics of the study of complete sets.

1 Introduction

Complete sets in a complexity class have been put forward as being 'the most difficult' in that class from a computational point of view (e.g. [WT89, LY88]). In some sense this is of course true since finding a simple algorithm for a complete set (under a polynomial time reduction) implies finding simple algorithms for any problem in the class. However, it is well-known that many classes contain problems that are 'harder' in the sense that they maintain greater distance from polynomial time computability than do the known complete sets for the same class. A complete set in a complexity class carries, because of the reductions of all the sets in the class to that complete set, all of the 'structure' that class might have. On the one hand, it is true that finding a simple algorithm for a complete set means that the structure of the class allows for simple algorithms, and that the complete set is thus the most difficult in that class. On the other hand, the requirement that the complete set must contain all of the structure of a complexity class guarantees that the complete set is not too weird in that class. The complete set that can represent any set in the class shows a kind of average behavior of the sets in that class. It is therefore just as true that complete sets may be committed to a simpler computational structure than other sets in the same class. For instance, on exponential time it is known that the complete sets under many-one reductions [Ber76, Ber77] have infinite P subsets, whereas the same class has an abundance of P-bi-immune sets [May94a]. Nonetheless complete sets can, because of their representational properties, tell us a lot about the complexity class they belong to, and the relation between complete sets can tell us a lot about the relation between complexity classes.

Many aspects and properties of complete sets can be studied. In this survey paper we want to review some research topics that are most interesting to us, not implying anything about the interest of related and unrelated aspects and properties. Many results on (the structure of) complete sets have appeared, and even survey papers have appeared on most of the topics surveyed in this paper. This has greatly simplified the task to which we have committed ourselves in this paper. Instead of having to explain an entire area from the ground up we can 'stand on the shoulders of giants' and view what new interesting discoveries have been made since these giants emerged. Sometimes of course, for historical reasons, we will wish to examine the origins of these giants and mention which results

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have initiated a particular field of interest.

The first aspect that aroused our own interest into completeness notions is the question whether completeness notions defined by different types of resource bounded reductions really are different. This topic might be called 'the structure of complete degrees' and was surveyed by Homer in [Hom90]. In Section 2, we review some of the developments that took place before and after that paper, trying as much as we can to complete the picture painted by that paper, and bring the matter up to our present day status of knowledge on that subject. The question whether all complete sets in a complexity class, especially under many-one reductions, but also under different types of reductions, really represent one and the same combinatorial problem has troubled the minds of computer scientists since it was first put forward by Berman and Hartmanis. Excellent and complete surveys on that subject have been written by Kurtz, Mahaney and Royer [KMR90] and Young [You90]. Very recently however, very interesting new developments have taken place in the isomorphism department, both where many-one and other (stronger) reductions are concerned. We wish to review these in Subsection 2.1

A question primarily related to the isomorphism problem, but also directly related to the inherent computational difficulty of complete sets, is that of density. Indeed the density of a set is one of the most apparent structural aspects of the set. Hemachandra, Ogiwara and Watanabe have surveyed this topic in [HOW92] with a strong focus on sets of very low density (sparse sets) and a very explicit disclaimer of the incompleteness of the survey. Maintaining the very same disclaimer, we wish to review, in Section 3, some recent results on the (non-)sparseness properties of complete sets in different classes.

An interesting property of sets first noted by Trakhtenbrot [Tra70], later explored by Ladner [Lad73] in recursion theory and transferred to the polynomial setting by Ambos-Spies [AS84] is that of mitoticity. Translating the question of mitoticity to complete sets boils then down to the question: "how many complete sets are really given to us?" Sometimes a complete set can, like a cell, split into two sets that remain complete. Sometimes it can't. The question of *mitoticity* and the related question of autoreducibility are addressed in Section 4.

The difficulty of a complete set, or any other set for that matter, does not tell anything about the hardness of individual instances of members of the set. To study this problem the concept of instance complexity was introduced by [KOSW86]. Some recent discoveries in that area for complete sets are surveyed in Section 5.

A very new way of looking at complete sets was introduced as a consequence of resource bounded measure introduced by Lutz [Lut90, Lut92]. New results and even a new notion of completeness, the 'weakly complete sets' are reviewed in Section 6.

Finally, in Section 7 we review investigations into robustness of complete sets. That is, which parts of complete sets are essential for these sets to remain complete. This particular property of complete sets is of course strongly related to the mitoticity property studied in Section 4.

We consider complete sets in this paper under reductions ranging from many-one via (forms of) truth-table to Turing reductions. Resource bounded—and in particular the polynomial time bounded—reductions are a central object of interest. Therefore, we will deliberately not review interesting, but less common completeness notions (as e.g.[BWSD77]), or reductions (as e.g.[Sel78]). Whenever we speak about particular reductions for which we do not explicitly state the resource bound or the non-existence thereof, we consider this bound to be polynomial time. Completeness notions are considered for complexity classes ranging from P to classes in the exponential hierarchy, but we will limit ourselves to language classes and therefore deliberately not review completeness in other types of classes (as e.g. function classes).

We do not wish to consume the space that is allowed for this paper with definitions and notations, which might also blur the reader's view on the structure we wish to discuss. Most definitions have appeared in the literature [LLS75] or in textbooks (cf. [BDG88]). An extensive overview of all sorts of resource bounded reductions can also be found in [Buh93]. We give other references when necessary.

2 Degrees of Complete Sets

For some reducibility r, a collection of sets that are interreducible (that is each pair of sets in this collection can be reduced to each other both ways) is called an r-degree. For any reduction there are of course many degrees. Within a complexity class the collection of complete sets under a given reduction forms a degree, with the nice additional feature that a membership proof usually consists of establishing reducibility in a single direction. The first most natural question to ask for the complete degrees of complexity classes defined by various reductions is whether the degrees defined by two different reductions are the same or not. As the standard reductions for a given resource bound are ordered by strength, the degrees are already ordered by inclusion, so the question is always whether one of the two collections is larger or not. If not, we speak of a collapse of the degree that was candidate for being larger.

The question of collapsing degrees is most interesting since on P all complete degrees collapse to the many-one complete degree, but *not* to the even smaller isomorphism degree. Showing the Turing complete (largest) degree different from the many-one complete (smallest) degree on any complexity class means therefore that this class is different from P. Unfortunately, until now this goal has only been achieved for complexity classes known to encompass superpolynomial deterministic classes [Wat87a, BHT91] (see also [Hom90]) and these we already knew to be different from P by the hierarchy theorems of Hartmanis and Stearns [HS65].

Authors nowadays are very reluctant to attack the P vs. NP question directly, but on the other hand they are very resourceful in finding conditions that are stronger than $P \neq NP$ from which to prove a property. A large part of the motivation for the introduction of the now celebrated P-selective sets (which we will henceforth call P-Selman sets) was to investigate the possibility of separating completeness notions on NP [Sel82]. Selman separated the reductions on NP on the assumption $E \neq NE \bigcap \text{co-}NE$, but not the completeness notions. In fact, to date, separation of the most distant (many-one and Turing) completeness notions on NP on an assumption $P \neq NP$ still fails.

The first results along these lines are the ones in [LY90], where they show that for each k there exist many-one NP-complete sets A and B such that $A \leq^P_{2-d} B$, where the reduction needs linear time, but A does not reduce many-one to B using less than n^k time. The most recent development in this field is the result of Lutz and Mayordomo [LM94]. From the strong assumption that NP forms a nonnegligible subset of EXP in the sense of resource bounded measure [Lut92] (see Section 6 for definitions), they prove that many-one and Turing completeness notions differ on NP. Since in this sense, P *is* a negligible subset of EXP, this assumption is stronger than $P \neq NP$.

Theorem 2.1 *If* $\mu(NP \mid EXP) \neq 0$ *then*

1. *The* $\leq^P_m$ *-complete degree is different from the* $\leq^P_{2-T}$ *-complete degree for NP [LM94].*
2. *The* $\leq^P_{2-tt}$ *-complete degree is different from the* $\leq^P_{2-T}$ *-complete degree for NP [May94b].*

This result was recently complemented by a result of Kautz and Miltersen [KM94], who show that relative to a random oracle (in the Lebesgue measure sense) NP is indeed not small (in the resource bounded measure sense). It would be interesting to improve the results in Theorem 2.1 to separate $\leq^P_{btt}$ -complete degree for NP from the $\leq^P_{tt}$ -complete degree, and the $\leq^P_{tt}$ -complete degree from the $\leq^P_T$ -complete degree.

On both deterministic [HKR93] and nondeterministic [BST93] exponential time the completeness notions induced by many-one and $\leq^P_{1-tt}$-reductions coincide. Using the techniques from the latter paper it can be shown [BG94] that the 1 truth-table complete degree on NP coincides with the many-one complete degree when NPSV-reductions or strong nondeterministic reductions (see for a definition [Lon82]) are used.

In [BST93], Buhrman, Spaan and Torenvliet also resolve the last remaining open questions from [Wat87a], left open in [BHT91]. In particular:

Theorem 2.2 ([BST93]) *For* $C \in \{E, EXP, NE, NEXP\}$

1. *for any* $k \geq 2$, $\leq^P_{k-c}$*-, and* $\leq^P_{k-d}$ *− completeness are incomparable on* C
2. *For any* k *and* l*, with* $k < l \leq 2^k - 2$, $\leq^P_{k-T}$*- and* $\leq^P_{l-tt}$*-completeness are incomparable on* C.

On the intermediate class $PSPACE$ so-far for only the two most distant reductions, $\leq^P_m$- and $\leq^P_T$-completeness notions have been separated [WT89] assuming that either randomized completeness notions differ or that $PSPACE$ has a set with a dense subset of high generalized Kolmogorov complexity. All other questions are open on $PSPACE$.

Some other open questions:

- Assume $\mu(NP \mid EXP) \neq 0$. Can the $\leq^P_{btt}$-complete degree for NP be separated from the $\leq^P_{tt}$-complete degree and the $\leq^P_{tt}$-complete degree from the $\leq^P_T$-complete degree?
- Can the complete degrees for NP under various reduction types be separated under the assumption $P \neq NP$?
- Does the $\leq^P_{1-tt}$-complete degree for NP collapse to the $\leq^P_m$-complete degree?

2.1 A Special Case: The Isomorphism Degrees

Berman and Hartmanis proved that many natural NP-complete problems are interreducible via polynomial-time computable isomorphisms. They conjectured that this held for all *NP*-complete problems. Their isomorphism proof was based on the paddability property of these problems rather than their completeness. Later, Joseph and Young defined a new type of *NP*-complete problems that seem much less likely to be isomorphic to SAT. Nowadays, the isomorphism conjecture is much less widely believed. Excellent overview papers [KMR90] and [You90] were written on this subject earlier.

At the time these papers were written there was already substantial evidence that the isomorphism conjecture as stated in [BH77] was not likely to hold. Most notably, Joseph and Young [JY85] proved that many-one reductions are flexible enough to define 'unnatural' complete problems in *NP*, which are not likely to be isomorphic to SAT. Also Kurtz, Mahaney and Royer [KMR89] showed that the isomorphism conjecture failed relative to a random oracle, and the only oracle support for the isomorphism conjecture was that of Goldsmith and Joseph [GJ86, Gol88] which was concerned with a weaker set of reductions. The celebrated Fenner, Fortnow, Kurtz [FFK92] oracle is the first full-blown relativized support of the conjecture. The isomorphism conjecture as well as it analagons for other complexity classes— the extended Berman-Hartmanis conjecture— is still very much open. For a treatment of the isomorphism conjecture for *EXP* see [Gan92]. Indeed, absolute results presently tend to go more in the direction of an isomorphism result for weaker type of reductions. First, Allender showed [All88] that *PSPACE* sets complete under 1-L reductions (A 1-L function is one that is computable by a logspace Turing machine that moves its input head only in one direction) are in fact polynomial time isomorphic, and set the stage for the investigation of the isomorphism conjecture under weaker reductions. In contrast to this, Burtschick and Hoene [BH92b] showed that on *PSPACE* sets complete under 1-L reductions are *not* necessarily isomorphic under 1-L isomorphisms. Recently, Agrawal and Biswas [AB93] closed the chapter started by Allender.

Theorem 2.3 ([AB93]) *Let C be a complexity class that is closed under lin-log reductions, e.g. P, NP, PSPACE. The sets complete for C under 1-L reductions are all p-isomorphic.*

In [Val82], Valiant defined the *projection*, an extremely low-level many-one reduction. Using this definition and a standard notion from logic for translating one theory into another, Immerman and Landau define a new reduction called *first order projection* [IL89]. In [ABI93] Allender, Balcázar and Immerman show that for the completeness notion induced by first order projections a very restricted version of the Berman Hartmanis conjecture holds.

Theorem 2.4 ([ABI93]) *Let C be a nice complexity class, e.g. P, NP, PSPACE. All sets complete for C under first order projections are isomorphic under first order isomorphisms.*

The completeness notion induced by first order projections is weaker than $\leq_m^P$ but incomparable to 1-L.

In conclusion, it seems that the final word on the isomorphism conjecture is not spoken, nor that it will be at any point in the near future. The evidence obtained so far seems to indicate that a subclass of *NP*-complete sets, complete under a reduction (much) stronger than $\leq_m^P$ may be isomorphic to SAT, but that there are sets in the $\leq_m^P$-degree of SAT that are not polynomial time isomorphic to this set. Finally, we note that even in the direction of stronger reducibilities the situation is not at all clear.

Joseph and Young, show that the k-creative sets that they introduce in [JY85] are *NP*-complete and furthermore, that *every* one-one polynomially computable and honest function is a productive function for some k-creative set. The (possible) existence of one-way functions would therefore make it *very hard* for k-creative sets to be isomorphic to SAT. (See also [Wan91].) Joseph and Young conjecture therefore that the Berman Hartmanis conjecture fails and because of the form of the evidence given in their paper their conjecture has been named the *encrypted complete set conjecture*. The encrypted complete set conjecture has also met with relativized counter examples [HH91]

Agrawal [Agr94] obtains results contradicting both the isomorphism and the encrypted complete set conjecture for some of the weaker reducibilities.

3 Density of Complete Sets

A first apparent structural property of a set is its density. Of the 2^n strings (on a binary alphabet) of length n available, how many are in the set and how many are not? For complete sets on *NP* this question became very important right after Berman and Hartmanis [BH77] put forward their conjecture on the isomorphism of *NP*-complete sets.

The recursive isomorphisms known to exist between r.e. complete sets put no restriction on the relative density of two r.e. complete sets. In fact, tally versions of (dense) r.e. complete sets are also r.e. complete and isomorphic to their original version or any other r.e. complete set by Myhill's theorem [Myh55]. For polynomial time computable isomorphisms the situation is completely different. A dense set (as all known natural NP-complete sets are) can *not* be polynomial time isomorphic to a sparse set. A major quest started right after the Berman Hartmanis paper came out was for an answer to the question: "Can sparse NP-complete sets exits?"

It was Mahaney [Mah82] who first gave a conditional (not unless $P = NP$) answer to this question and since the conclusion is widely disbelieved, the question for many-one NP-complete sets is assumed to be settled. For other completeness notions, for other complexity classes, and for other subexponential densities, the problem is still very much open. The recent paper [HOW92] of Hemachandra, Ogiwara and Watanabe has in its Section 3 a very thorough survey of the (im)possibilities of sparse complete sets for P, NP and other classes.

Since then, attention has shifted to the exponential time classes. One of the reasons is that there is a connection between complexity classes having $\leq_T^P$ -hard (or complete) sets with small density and these complexity classes having polynomial size circuits. Hence proving that the $\leq_T^P$ -complete sets for EXP are not sparse implies that EXP is not included in $P/poly$. This however is a major open problem. Another reason perhaps is that conditional results—i.e. unless $P = NP$—about NP complete and hard sets usually translate upwards to absolute results about E and EXP (see the proof of Corollary 5.8, Section 5). Therefore, it is natural to investigate the density of complete and hard sets for E and EXP directly. Research along these lines was initiated by Watanabe [Wat87b]. We mention here only the most recent results.

Theorem 3.1 ([Fu93])

1. *For $\alpha < 1$, all $\leq_{n^\alpha-T}^P$ -hard sets for EXP are exponentially dense.*
2. *For $\alpha < \frac{1}{4}$, all $\leq_{n^\alpha-T}^P$ -hard sets for E are exponentially dense.*

Lutz and Mayordomo[LM93], led by the idea that it is easier to show the majority of E not in $P/poly$ than one particular set, have the following theorem, that is incomparable with the previous one.

Theorem 3.2 ([LM93]) *For $\alpha < 1$, all $\leq_{n^\alpha-tt}^P$ -weakly-hard sets for E and EXP are exponentially dense.*

These theorems seem to indicate that we are close to proving that EXP does not have polynomial size circuits. However, all the proof techniques used so far relativize and there are oracles relative to which EXP has polynomial size circuits [Wil85]. Finally, we want to mention that Homer and Mocas [HM93] attacked the problem of EXP having polynomial size circuits from yet another angle. They show that EXP is not included in *advice classes* having advice that is bounded in size by a *fixed* polynomial.

Theorem 3.3 ([HM93]) *for every k there exists a set A in EXP such that A is not in $DTIME(2^{n^k})/n^k$.*

Improving upon these results is very interesting but also seems very hard. There seems to be some room for improving upon Theorem 3.1 and Theorem 3.2 using non-relativizing techniques. More specific:

- For any k: are the $\leq_{n^k-T}^P$ complete sets for EXP exponentially dense?
- For any k: are the $\leq_{n^k-T}^P$ weakly complete sets for EXP exponentially dense?
- Related to [BH92a], are the $\leq_T^P$-complete sets for $NEXP$ not sparse unless $NEXP = \Sigma_2^P$?

4 Mitoticity and Auto-Reducibility

In this section we will look at the internal structure of the complete sets. In recursion theory many structural properties of sets have been successfully studied. One of these notions, introduced by Trakhtenbrot [Tra70], is autoreducibility. A set A is autoreducible if A can be recognized by an (in our setting polynomial time) oracle Turing machine that on input x never queries x, using oracle A. Hence autoreducible sets are sets that contain a redundancy of information, since the information $A(x)$ is present among the strings different from x. Trakhtenbrot [Tra70] noticed that every $\leq_m$-degree contains an autoreducible set for $A \oplus A$ is[1] autoreducible for every A.

The redundancy aspect of autoreducible sets becomes even more apparent if we compare this notion with another notion, called mitoticity. This notion, introduced by Ladner [Lad73], makes precise the aspects of informational redundancy in an r.e. set. An r.e. set A is called mitotic if A can be divided into two disjoint r.e. subsets A_0 and A_1, such that $A \equiv_T A_0 \equiv_T A_1$. The term mitotic is appropriate because if A is mitotic then it truly consists out of two parts that each

[1] Recall that $A \oplus A$ is the marked union of A and A.

contain the same amount of information as the original set, as is the case in biology where the mitosis indicates the splitting of a cell into two cells that both contain the same information stored in the DNA of the original cell. Ladner showed, surprisingly, that the two seemingly different notions—autoreducibility and mitoticity—coincide for r.e. sets.

Ambos-Spies, led by the idea that recursion theoretical notions help understand the world of complexity theory, was the first to carry over the two notions to a polynomial time bounded setting [AS84]. The autoreducibility notion easily translates into:

Definition 4.1 A set A is polynomially autoreducible (autoreducible for short) if there exists a polynomial time oracle Turing machine M such that:

1. $A = L(M, A)$.
2. for all x: $x \notin Q(M, x, A)$.

Where $Q(M, x, A)$ is the set of queries M generates on input x with A as an oracle.

We say that a set A is r-autoreducible if the machine M in fact is a r-reduction from A to A. Note that the randomized version of autoreducibility, studied in [Yao90, BF92], is called *coherence*.

Translating the notion of mitoticity to a polynomial time setting is less obvious. In [AS84] two definitions are given.

Definition 4.2

1. A recursive set A is polynomial time m(T)-mitotic (m(T)-mitotic for short) if there exists a set $B \in P$ such that: $A \equiv^P_{m(T)} A \bigcap B \equiv^P_{m(T)} A \bigcap \overline{B}$.
2. A recursive set A is polynomial time weakly m(T)-mitotic (weakly m(T)-mitotic for short) if there exists disjoint sets A_0 and A_1 such that: $A = A_0 \bigcup A_1$ and $A \equiv^P_{m(T)} A_0 \equiv^P_{m(T)} A_1$.

In [AS84] it is shown that the two notions of mitoticity differ quite a bit. Note however that mitotic implies weakly-mitotic. In the same paper the relation between autoreducibility and mitoticity is investigated. It turns out that these notions are not equivalent as in the recursive setting:

Theorem 4.3 ([AS84])

1. *For any A, if A is m(T)-mitotic then A is m(T)-autoreducible.*
2. *On recursive tally sets m-mitoticity and m-autoreducibility are the same.*
3. *There exists a weakly m-mitotic set that is not m-autoreducible.*
4. *There exists a tally set which is T-autoreducible but not T-mitotic.*
5. *There exists a weakly T-mitotic set that is not T-autoreducible.*

The questions whether

- m-autoreducibility implies m-mitoticity, or
- m-autoreducibility implies weakly m-mitoticity, or
- T-autoreducibility implies weakly T-mitoticity,

remain open. In the following we will turn back to the complete sets.

4.1 Completeness

The question we are interested in is whether compete sets are autoreducible and/or mitotic. Before we look at this question we want to mention the situation in recursion theory. Recall that mitoticity and autoreducibility coincide for r.e. sets. Ladner [Lad73] showed that not all $\leq_T$-complete sets for r.e. are autoreducible and hence not all are mitotic. Since the notions—mitoticity and autoreducibility—are not a priori the same for complete sets we will treat them separately.

4.1.1 Mitoticity

Starting with *NP*, we note that all natural *NP* complete sets are m-mitotic. The reason is that they are all p-isomorphic to SAT, and it can be easily shown that SAT is m-mitotic. Moreover all the notions of mitoticity and autoreducibility are invariant under p-isomorphisms. Thus it follows that if the isomorphism conjecture is true, then all $\leq^P_m$ -complete sets for *NP* are m-mitotic (and hence weakly m-mitotic). Similar observations are valid for all levels of the Polynomial Hierarchy and *PSPACE*. However it is open whether any complete degree (under some reduction) for these classes is completely (weakly) m(T)-mitotic.

For *EXP* a little bit more can be said:

Theorem 4.4 ([BHT93]) *All $\leq^P_m$ -complete sets for EXP are weakly m-mitotic.*

This theorem parallels nicely the situation for r.e. sets. On the other hand it is shown also that not all $\leq_{tt}^{P}$-complete sets are weakly m-mitotic.

Theorem 4.5 ([BHT93]) *There exists a $\leq_{3-tt}^{P}$-complete set for EXP that is not weakly m-mitotic.*

For *NEXP* the situation is even less clear. It can be shown that every $\leq_{m}^{P}$-complete set can be split into *infinitely* many disjoint subsets, such that each of these subsets are $\equiv_{m}^{P}$ to A [BHT93], but nothing is known for the mitotic case. Specific open problems are:

- Are all $\leq_{m}^{P}$-complete sets for *EXP* m-mitotic?
- Are all $\leq_{m}^{P}$-complete sets for *NP*, *PH*, *PSPACE*, or *NEXP* (weakly) m(T)-mitotic?
- Are all $\leq_{T}^{P}$-complete sets for *NP*, *PH*, *PSPACE*, *EXP* or *NEXP* (weakly) m(T)-mitotic?

4.1.2 Autoreducibility

The situation with respect to autoreducibility is somewhat better understood. Surprisingly, the parallel with recursion theory disappears with respect to complete sets. Of course the same remarks about $\leq_{m}^{P}$-complete sets that were made for mitoticity are true for autoreducibility but the $\leq_{T}^{P}$-complete sets seem to behave differently.

Theorem 4.6 ([BF92, BvHT93]) *Every $\leq_{T}^{P}$-complete set for NP is autoreducible.*

In fact in [BF92] it is shown that all $\leq_{T}^{P}$-degrees that contain a self-reducible set are completely autoreducible hence:

Theorem 4.7 ([BF92]) *All $\leq_{T}^{P}$-complete sets for all levels of the Polynomial Hierarchy and PSPACE are autoreducible.*

Unfortunately the techniques in [BF92] only apply to sets within *PSPACE*. Extending the techniques in [BvHT93] it can be shown that also the complete sets for *EXP* are autoreducible.

Theorem 4.8 ([BFL94]) *All $\leq_{T}^{P}$-complete sets for EXP are autoreducible.*

It can also be shown that all $\leq_{2-tt}^{P}$-complete sets for *EXP* are 2-tt-autoreducible. Furthermore these results are optimal in some sense:

Theorem 4.9 ([BFL94])

1. *For all $k > 0$, there exists a set that is $\leq_{k-tt}^{P}$-complete for EXP that is not (k-1)-tt-autoreducible.*
2. *There exists a set that is $\leq_{tt}^{P}$-complete that is not btt-autoreducible.*
3. *There exists a set that is $\leq_{T}^{P}$-complete but not tt-autoreducible.*

Questions that remain:

- Are all $\leq_{m}^{P}$-complete sets for *NP* or *PSPACE* m-autoreducible?
- Are all $\leq_{tt}^{P}$-complete sets for *NP*, *PSPACE* or *EXP* *tt*-autoreducible?
- Are all $\leq_{T}^{P}$-complete sets for *NEXP* autoreducible?

5 Instance Complexity of Complete Sets

Computational intractability of decision problems can be divided into two areas of research. One approach, dubbed 'distributional' by some people, is to study the way the elements or instances are distributed over the problem and the complement of the problem. A problem is difficult if the instances are distributed in a very irregular manner, since feasible algorithms can only detect simple distributions. This line of research is probably the most common one in complexity theory. This approach expresses the complexity of a problem in the asymptotic behavior of the algorithm for solving the problem. The other line of research consists of expressing the complexity of a *single* instance as opposed to the asymptotical analysis. An instance is hard with respect to this point of view if it is a hard instance, independent of any algorithm for solving the problem. Examples of this approach are the study of *complexity cores* [Lyn75, OS86, BD88] and *instance complexity* [KOSW86, OKSW94]. In this section we want to discuss the latter and its relation with complete sets.

5.1 Definitions and Facts

The problem with asymptotic complexity analysis is that finite sets—and hence single instances—have low complexity and thus are feasible. The reason for this is that any finite set can be stored in the finite program of the computational device. The finite set can then be recognized using table lookup in constant time. To overcome this problem the *size* of the program is taken into account as well.

Consider the class of Turing machines that on each input always output 1 (accept) or 0 (reject) or ? (don't know). Such a Turing machine is said to be *consistent* with a set A iff, for all inputs x such that $M(x) \neq ?$, $M(x) = A(x)$. The t-bounded instance complexity of an instance x with respect to a set A, $\mathrm{ic}^t(x : A)$, is the size of the smallest—in length—Turing machine M, such that:

1. M is consistent with A.
2. M runs in time bound t for all inputs.
3. $M(x) \neq ?$.

Note that the above definition is ambiguous since the notion "the size of the smallest Turing machine" is not well defined. In order to get a correct definition one needs to define the notion relative to programs of some interpreter. Doing so, it can be shown that there exist a universal Turing machine U such that the t-bounded instance complexity relative to U differs no more than an additive constant factor from the t'-bounded instance complexity relative to any interpreter, where $t(n)' = ct(n)\log(t(n)) + c$. For details see [OKSW94] or [BO94] in these proceedings. The above (incomplete) definition however will fit our purposes here.

Note also that the instance complexity is closely related to the time bounded Kolmogorov Complexity [Har83, LV93]. Recall that the t-bounded Kolmogorov complexity of a string x, $K^t(x)$, is the size of the smallest Turing machine M such that:

1. $M(\lambda)$ outputs x.
2. M takes no more than $t(|x|)$ steps.

Again this definition has the same problem as the previous one and can be phrased robustly relative to a universal Turing machine. The difference between the Kolmogorov complexity and the instance complexity is that the Kolmogorov complexity for one particular instance x is fixed, whereas the instance complexity differs with respect to different sets A. There is however a simple relationship between the Kolmogorov complexity and the instance complexity:

Proposition 5.1 ([OKSW94]) *For any time constructible function t, there exists a constant c such that for any set A and string x,*

$$\mathrm{ic}^{t'}(x : A) \leq K^t(x) + c,$$

where $t'(n) = ct(n)\log(t(n)) + c$.

The proposition states that the Kolmogorov complexity always is an upper bound on the instance complexity. On the other hand, we are interested in saying that a set A has instances that are hard or difficult. A natural way of expressing this is as follows:

Definition 5.2 A set A has t-hard instances iff there exists a constant c such that for infinitely many instances x, $\mathrm{ic}^t(x : A) \geq K^t(x) - c$.

In this definition we did not consider the $\log(n)$ factor that comes out of Proposition 5.1. Only considering polynomial instance complexity and admitting this $(\log(n))$ factor we also have the following notion of hard instances:

Definition 5.3 A set A has P-hard instances if for any polynomial t there exist a polynomial t' and a constant c such that for infinitely many x, $\mathrm{ic}^t(x : A) \geq K^{t'}(x) - c$.

In [OKSW94] it was conjectured that any set not recognizable in a certain time bound t will have t-hard instances.

Conjecture 5.4 ([OKSW94]) *Let A be a set not in $DTIME(t)$. Then there exist infinitely many x and a constant c such that $\mathrm{ic}^t(x : A) \geq K^t(x) - c$.*

As evidence for their conjecture it is proved for any set A in $E \setminus P$:

Theorem 5.5 ([OKSW94]) *Let A be a set in $E \setminus P$. There exists a constant c such that for any polynomial t there exists a constant c_t and infinitely many x, such that, $\mathrm{ic}^t(x : A) \geq K^{2^{cn}}(x) - c_t$.*

More recently, Kummer gave more evidence for this conjecture in a very nice paper:

Theorem 5.6 ([Kum94]) *Let A be a set not in E. Then for any c there exists a c' and d such that for infinitely many x, $\mathrm{ic}^{2^{cn}}(x : A) \geq K^{2^{c'n}}(x) - d$.*

We will now shift our attention to complete sets.

5.2 Complete sets and instance complexity

The natural question addressed here is: "Do complete sets have hard instances?" Of course for classes like *NP* it is important to fix the time bound of the instance complexity to polynomial. In this section we will therefore only consider $p(n)$-bounded instance complexity for $p(n)$ some polynomial.

The first partial results along these lines were obtained by Orponen in [Orp90]. In this paper it was shown that it cannot be the case that the instance complexity of $\leq_{btt}^{p}$-hard sets for *NP* is low for all the instances unless $P = NP$.

Theorem 5.7 ([Orp90]) *If A is self-reducible and there exist a constant c and a polynomial t such that for all x, $\mathrm{ic}^t(x : A) \leq c\log(|x|) + c$ then A is in P.*

It now follows, using the fact that the instance complexity can not decrease much through a $\leq_{btt}^{P}$-reduction, that all classes that possess self-reducible Turing complete sets—such as *PSPACE* and *NP*—cannot have $\leq_{btt}^{P}$-hard sets with low instance complexity everywhere unless they are equal to P.

The above theorem yields as an easy corollary an absolute result about *EXP*. Later on we will see that this can be improved.

Corollary 5.8 *There does not exists a $\leq_{btt}^{P}$-hard set A for EXP such that for all x, $\mathrm{ic}^t(x : A) \leq c\log(|x|) + c$ for some constant c and polynomial t.*

Proof. Note that sets with low instance complexity everywhere, as in the corollary, are in $P/poly$. Assume for a contradiction that the corollary is not true, it now follows by a well-known theorem of Karp and Lipton [KL80] that if *EXP* has a $\leq_{btt}^{P}$ -hard set that is sparse, that then $EXP = \Sigma_2^P$. Furthermore, Σ_2^P contains self-reducible Turing complete sets and hence by Theorem 5.7 above it follows that $\Sigma_2^P = P$, but now we have a contradiction: $EXP = \Sigma_2^P = P$. □

The above theorem and corollary are still far away from the instance complexity conjecture. Shifting to complete sets however it is shown in [BO94] that the conjecture is correct for $\leq_m^P$-complete sets for E and polynomial time bounds t.

Theorem 5.9 ([BO94]) *Let A be a $\leq_m^P$-complete set for E. Then there exists an exponentially dense set $C \subseteq A$, such that for all $x \in C : \mathrm{ic}^t(x : A) \geq K^t(x) - c$, for c some constant and $t \in \omega(n\log(n))$.*

Note that the theorem is also true for *EXP* and $\leq_{1-tt}^{P}$-reductions. The theorem states that there is an exponentially dense subset of any complete set for E that has hard instances. This is in contrast with the fact that there also exists a subset of A that is in P [Ber77] [2] and hence has constant instance complexity for all x. Note also that Theorem 5.9 implies that any set B that is $\leq_{btt}^{P}$-hard for E contains infinitely many instances x such that $\mathrm{ic}^t(x : B) \geq cK^t(x) - d$ for some constants c and d. It remains open however whether Theorem 5.9 can be proven for complete sets with respect to other reductions. It even remains open if we drop the density requirement.

It is not known whether Theorem 5.9 is true for $\leq_m^P$-complete sets for *NP*. However, Orponen showed in [Orp90] that under the assumption that $E \neq NE$ the *NP*-hard sets have *P*-hard instances.

Theorem 5.10 ([Orp90]) *If $NE \neq E$ then all $\leq_{1-tt}^{P}$-hard sets for NP have P-hard instances.*

Specific open problems:

1. If A is complete for *EXP* under $\leq_{btt}^{p}$, $\leq_{tt}^{P}$ or $\leq_T^P$, does A have t-hard instances, for t a polynomial?
2. If A is complete for *EXP* under $\leq_{tt}^{P}$ or $\leq_T^P$, does A have p-hard instances?
3. Is the set of hard instances in 1 or 2 dense?
4. Do many-one complete sets for *NP* have t-hard instances?
5. Related to [HW94], do many-one complete sets for *EXP* have a dense subset that has constant instance complexity everywhere?
6. Do many-one complete sets for *NEXP* have an infinite (dense) subset that has constant instance complexity everywhere?

6 Resource Bounded Measure and Complete Sets

In classical Lebesgue measure theory, the *measure* of a class of sets in the class of all sets is defined as the probability that a random experiment builds a language in this class. The random experiment consists of the tosses of a fair coin that decide for each of the strings of Σ^* in turn whether it is in the language or not. If the probability of success for such an experiment is 1 (i.e. if the probability that the language built is not in the class is 0) we say that such a class has *measure* 1. An often quoted result in this area is that of Bennett and Gill [BG81] who proved that the class of sets that, when used as an oracle to separate P from NP has measure 1 in the class of all sets.

Measure theory and probability theory in the classical sense have one major drawback. The classes of languages we are most interested in in complexity theory are mostly recursively presentable, which means

[2] See for an extensive study on immunity and $\leq_m^P$ complete sets for *EXP* and *NEXP* [HW94].

roughly, definable by an enumeration of programs. Therefore, any such class consists of a countable number of sets. In classical Lebesgue theory any countable set has measure 0. In his thesis [Lut87], Lutz started to remedy this situation by defining *resource bounded measure* as a generalization of Lebesgue measure. He defines the measure of a class of sets in relation to the class of exponential time (or exponential space) computable sets. The measure thus obtained for a class of sets C *in* EXP gives a notion for the relative size of classes w.r.t. the size of EXP.

Crucial to the concept of resource bounded measure is the concept of a *density* system, which in more recent work of Lutz [Lut94] is presented with the intuitively more appealing notion of a *martingale*. First, languages are identified with strings in $\{0,1\}^\infty$, their *characteristic* strings.

Definition 6.1 A *martingale* is a function $d : \{0,1\}^* \mapsto [0,\infty)$ with the property that for all $w \in \{0,1\}^*$:

$$d(w) = \frac{d(w0) + d(w1)}{2}$$

A martingale *succeeds on a language* $A \subseteq \{0,1\}^*$ if

$$\limsup_{n \mapsto \infty} d(\chi_A[0...n-1]) = \infty$$

For each martingale d, define the set $S^\infty[d] = \{A \subseteq \{0,1\}^* \mid d \text{ succeeds on } A\}$.

Martingales are real-valued functions. Therefore, their computations must employ finite approximations of real numbers. For this purpose we consider the set of *dyadic rationals* $D = \{m.2^{-n} \mid m,n \in \mathbb{N}\}$. Then we can define *computations* of a martingale d as a function $\hat{d} : \mathbb{N} \times \{0,1\}^* \mapsto D$ with the property that $|\hat{d}(r,w) - d(w)| \leq 2^{-r}$ for all r and all w. For such an approximation of a martingale we can discuss its complexity, e.g. we can say that such an approximation is a p-computation if it is polynomial time computable. A martingale for which a p-computation exists is then called a p-martingale.

Now we can say:

Definition 6.2

1. A set X of languages has p-measure 0 (written $\mu_p(X) = 0$) if there is a p-martingale d such that $X \subseteq S^\infty[d]$.
2. A set X of languages has *measure* 0 in E (written $\mu(X \mid E) = 0$) if $\mu_p(X \cap E) = 0$.
3. A set X of languages has *measure* 1 in E (written $\mu(X \mid E) = 1$) if $\mu(X^c \mid E) = 0$.

Using martingales that are allowed more time, one can define an appropriate measure in EXP. As it turns out the entire class E does not have measure 0 in E ($\mu(E \mid E) \neq 0$), whereas the class P has measure 0 in E, which is most fortunate since a proof that some class of sets does *not* have measure 0 in E is then a proof that this class is not contained in P. For a more complete view on resource bounded measure we refer the reader to the work of Lutz, especially to [Lut92] and to Mayordomo's thesis [May94b].

Most interesting in this section is the recent discovery of Lutz that defines a new completeness notion for exponential time that could not have existed without the definition of resource bounded measure. In the standard definition of (many-one) completeness for exponential time, *all* sets in E are reducible to a complete set. Lutz proposes in [Lut90] a weaker form of completeness that only requires that a nonnegligible subclass (i.e. a subclass that does not have measure 0 in E) be reducible to the complete set. Lutz dubs this new form of completeness *weakly completeness* for E and it seems that this weakly completeness is indeed a new and rich notion which is interesting and worth investigating.

Until recently, it was not known whether weakly complete sets and many-one complete sets coincide on EXP. In [Lut94] however, it is shown that there indeed exists a language that is weakly $\leq_m^P$-complete, but not $\leq_m^P$-complete for E. Ambos-Spies, Terwijn and Zheng [ASTZ94], by using a new approach to p-measure-1 results, show that weakly complete sets are abundant in E and show that there are weakly complete sets that are not even $\leq_{btt}^P$-complete in E.

Juedes and Lutz [JL94] go on to show that the weakly complete sets are even more different from $\leq_m^P$-complete sets. It is known, by simple padding arguments, that a language C is $\leq_m^P$-complete for E if and only if it is in E and it is $\leq_m^P$-complete for EXP. For weakly complete sets the situation is however different.

Theorem 6.3 ([JL94])

1. *Every language that is weakly $\leq_m^P$-complete for E is weakly $\leq_m^P$-complete for EXP.*
2. *There is a language in E that is weakly $\leq_m^P$-complete for EXP, but not for E.*

We already mentioned the recent result of Lutz and Mayordomo on the inequality of completeness notions on NP, provided NP does not have p-measure 0, in our section on degrees of complete sets (Section 2).

Another interesting result, pertaining to more standard completeness notions, is that some complete degrees for E have measure 0.

Theorem 6.4

1. *The $\leq_m^P$ -complete degree for E has measure 0 in E [JL93, May94a].*

2. *For all k, the $\leq_{k-tt}^P$ -complete degree for E has measure 0 in E [BM94].*

3. *The $\leq_{btt}^P$ -complete degree for E has measure 0 in E [ASNT94].*

On other complexity classes than exponential time, little is known on the subject of measure and completeness. This is in part because defining a resource bounded measure on subexponential classes is not a trivial job. The definition of measure heavily depends on the computation of characteristic sequences. Since for length n the characteristic sequence of all strings of length less than or equal to n is of exponential size, this takes at least exponential time. In [May92] a measure on $PSPACE$ is defined. In [May94b] it is shown that the $\leq_m^{logspace}$ -complete degree has measure 0 in $PSPACE$.

Very recently, Allender and Strauss [AS94] attempted to define a resource bounded measure usable even on P. This is however still very much 'work in progress'.

Some open problems in this rapidly evolving field are:

- Do the $\leq_{tt}^P$-complete sets for E and EXP have measure 0 in E and EXP?

- Do the weakly $\leq_m^P$-complete sets for E or EXP coincide with the weakly $\leq_{1-tt}^P$ complete sets for E or EXP?

- Can the weakly completeness notions under various reduction types be separated for E or EXP?

7 Stability of Complete Sets

Any complexity class that allows a diagonal like [Ber77] or [GH88] has the property that its $\leq_m^P$-complete sets are complete under 1-1 reductions, and it has no sparse $\leq_m^P$-complete set. One can then play the following game with complete sets. Suppose I were to take out some elements of the set. How many elements can be taken away without losing completeness? Or, what is the density and/or the complexity of the set to be taken out of the complete set that makes it lose its completeness? The question seems a kind of natural counterpart to the question treated in Section 4. There, we established that the 'completeness information' in many sorts of complete sets is so abundantly present, that sets can be taken apart in various ways without any of the parts giving up completeness. Here we will see that the completeness information may be stored in *very* sparse subsets of the complete set.

In his paper [Sch86], Schöning generalized and improved upon the work of Yesha [Yes83] and the already quoted non-sparseness result of Berman [Ber77] and showed that for complete sets A in EXP and every set D in P, the set $A\Delta D$ is of exponential density. In [TFL93], Tang, Fu and Liu showed that even for subexponential time computable D, the set $A\Delta D$ remains exponential time complete. They go on to show that for an arbitrary sparseness condition, there exists a single subexponential time computable set S, such that for *any* exponential time complete set A, the set $A-S$ is no longer exponential time complete. Their proof hinges on another old result of [Ber77] that states that the complete sets in EXP are complete under length increasing reductions. The subexponential time computable set is constructed by chosing a sufficiently sparse subset of $\{0\}^*$, and defining $S=\{0^{b_i} : |f_i(0^{b_i})| > b_i\}$, where the b_i are sufficiently far apart. Then there can be no length increasing $\leq_m^P$-reduction from $\{0\}^*$ to $A-S$, since for any length increasing function at least one of the images of a string in $\{0\}^*$ is not in $A-S$. Using the fact that $\leq_m^P$-hard sets for exponential time are also hard under 1-1 reductions, the result for complete sets can (and has in [TFL93]) be extended to EXP hard sets.

The natural question to ask next after the result was obtained for many-one reductions is: "How do complete sets under weaker types of reductions behave?" The answer to this question was given to a large extent in [BHT93]. They show indeed that the observation of [TFL93] about this structural aspect of complete sets in exponential time is not limited to many-one completeness.

Theorem 7.1
Given a recursive non-decreasing function $g(n)$ with $\lim_{n\mapsto\infty} g(n)=\infty$. There exists $g(n)$-sparse subexponential time computable sets S_1, S_2, S_3 and S_4 such that: [TFL93] For any $\leq_m^P$-hard set A, the set $A-S$ is not $\leq_m^P$-hard, and [BHT93]

1. *For any $\leq_{btt}^P$-complete set A for EXP the set $A - S_1$ is not $\leq_{btt}^P$-hard.*

2. *For any $\leq_c^P$-hard set A for EXP the set $A - S_2$ is not $\leq_c^P$-hard.*

3. *For any $\leq_d^P$-hard set A for EXP the set $A \cup S_3$ is not $\leq_d^P$-hard.*

Not only the sparseness of the sets S_i is controllable, but also their 'subexponentiality'. By slowing down the diagonalization over the different reductions, the computation of the image of the reduction can be brought arbitrarily close to polynomial time. Tang, Fu and Liu [TFL93] wondered what limited this closeness for $\leq_m^P$-reductions. We readdressed the question [BHT93] for weaker reductions. The answer was in all cases that it exactly breaks off at polynomial time.

Theorem 7.2 ([TFL93],[BHT93]) *For any set A that is $\leq_m^P$-, $\leq_c^P$-, $\leq_d^P$-, or $\leq_{2-tt}^P$-hard for EXP and any $p(n)$ sparse polynomial time computable set S, the set $A - S$ remains hard w.r.t. the same reduction.*

The last ($\leq_{2-tt}^P$) result was not extended to k-tt or larger truth tables mostly because of the rapid increase in the complexity of the proof. There is strong evidence that the line of proof must break down somewhere between $\leq_{2-tt}^P$ and $\leq_{tt}^P$-completeness, since the (non)existence of a sparse, Turing complete set is still a very hard open problem, and of course existence of such a complete set would imply the existence of a tally complete set in EXP, which has a very easy polynomial time computable sparse (since tally) superset.

Open problems:

- Let A be $\leq_{tt}^P$ or $\leq_T^P$-complete for EXP. Does there exist a sparse subexponential time computable set S such that $A - S$ is not complete?

- Let A be $\leq_{btt}^P$-complete for EXP. Is for every sparse $S \in P$, $A - S$ still $\leq_{btt}^P$-complete.

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